

QUASI-PRÜFER EXTENSIONS OF RINGS

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ABSTRACT. We introduce quasi-Prüfer ring extensions, in order to relativize quasi-Prüfer domains and to take also into account some contexts in recent papers, where such extensions appear in a hidden form. An extension is quasi-Prüfer if and only if it is an INC pair. The class of these extensions has nice stability properties. We also define almost-Prüfer extensions that are quasi-Prüfer, the converse being not true. Quasi-Prüfer extensions are closely linked to finiteness properties of fibers. Applications are given for FMC extensions, because they are quasi-Prüfer.

1. INTRODUCTION AND NOTATION

We consider the category of commutative and unital rings. An epimorphism is an epimorphism of this category. Let $R \subseteq S$ be a (ring) extension. The set of all R -subalgebras of S is denoted by $[R, S]$. The extension $R \subseteq S$ is said to have FIP (for the “finitely many intermediate algebras property”) if $[R, S]$ is finite. A *chain* of R -subalgebras of S is a set of elements of $[R, S]$ that are pairwise comparable with respect to inclusion. We say that the extension $R \subseteq S$ has FCP (for the “finite chain property”) if each chain in $[R, S]$ is finite. Dobbs and the authors characterized FCP and FIP extensions [10]. Clearly, an extension that satisfies FIP must also satisfy FCP. An extension $R \subseteq S$ is called FMC if there is a finite maximal chain of extensions from R to S .

We begin by explaining our motivations and aims. The reader who is not familiar with the notions used will find some Scholia in the sequel, as well as necessary definitions that exist in the literature. Knebusch and Zang introduced Prüfer extensions in their book [26]. Actually, these extensions are nothing but normal pairs, that are intensively studied in the literature. We do not intend to give an extensive list of recent papers, written by Ayache, Ben Nasr, Dobbs, Jaballah, Jarbouh and some others. We are indebted to these authors because their papers are a rich source of suggestions. We observed that some of them

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are dealing with FCP (FIP, FMC) extensions, followed by a Prüfer extension, perhaps under a hidden form. These extensions reminded us quasi-Prüfer domains (see [18]). Therefore, we introduced in [38] *quasi-Prüfer* extensions $R \subseteq S$ as extensions that can be factored $R \subseteq R' \subseteq S$, where the first extension is integral and the second is Prüfer. Note that FMC extensions are quasi-Prüfer.

We give a systematic study of quasi-Prüfer extensions in Section 2 and Section 3. The class of quasi-Prüfer extensions has a nice behavior with respect to the classical operations of commutative algebra. An important result is that quasi-Prüfer extensions coincide with INC-pairs. Another one is that this class is stable under forming subextensions and composition. A striking result is the stability of the class of quasi-Prüfer extensions by absolutely flat base change, like localizations and Henselizations. Any ring extension $R \subseteq S$ admits a quasi-Prüfer closure, contained in S . Examples are provided by Laskerian pairs, open pairs and the pseudo-Prüfer pairs of Dobbs-Shapiro [15].

Section 4 deals with *almost-Prüfer* extensions, a special kind of quasi-Prüfer extensions. They are of the form $R \subseteq T \subseteq S$, where the first extension is Prüfer and the second is integral. Any ring extension admits an almost-Prüfer closure, contained in S . The class of almost-Prüfer extensions seems to have less properties than the class of quasi-Prüfer extensions but has the advantage of the commutation of Prüfer closures with localizations at prime ideals. We examine the transfer of the quasi (almost)-Prüfer properties to subextensions.

Section 5 study the transfer of the quasi (almost)-Prüfer properties to Nagata extensions.

In section 6, we complete and generalize the results of Ayache-Dobbs in [5], with respect to the finiteness of fibers. These authors have evidently considered particular cases of quasi-Prüfer extensions. A main result is that if $R \subseteq S$ is quasi-Prüfer with finite fibers, then so is $R \subseteq T$ for $T \in [R, S]$. In particular, we recover a result of [5] about FMC extensions.

Now Section 7 gives calculations of $[[R, S]]$ with respect to its Prüfer closure, quasi-Prüfer (almost-Prüfer) closure in case $R \subseteq S$ has FCP.

1.1. Recalls about some results and definitions. The reader is warned that we will mostly use the definition of Prüfer extensions by flat epimorphic subextensions investigated in [26]. The results needed may be found in Scholium A for flat epimorphic extensions and some results of [26] are summarized in Scholium B. Their powers give quick proofs of results that are generalizations of results of the literature.

As long as FCP or FMC extensions are concerned, we use minimal (ring) extensions, a concept introduced by Ferrand-Olivier [17]. An extension $R \subset S$ is called *minimal* if $[R, S] = \{R, S\}$. It is known that a minimal extension is either module-finite or a flat epimorphism [17] and these conditions are mutually exclusive. There are three types of integral minimal (module-finite) extensions: ramified, decomposed or inert [36, Theorem 3.3]. A minimal extension $R \subset S$ admits a crucial ideal $\mathcal{C}(R, S) =: M$ which is maximal in R and such that $R_P = S_P$ for each $P \neq M, P \in \text{Spec}(R)$. Moreover, $\mathcal{C}(R, S) = (R : S)$ when $R \subset S$ is an integral minimal extension. The key connection between the above ideas is that if $R \subseteq S$ has FCP or FMC, then any maximal (necessarily finite) chain of R -subalgebras of S , $R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$, with *length* $n < \infty$, results from juxtaposing n minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq n-1$.

Following [24], we define the *length* $\ell[R, S]$ of $[R, S]$ as the supremum of the lengths of chains in $[R, S]$. In particular, if $\ell[R, S] = r$, for some integer r , there exists a maximal chain in $[R, S]$ with length r .

As usual, $\text{Spec}(R)$, $\text{Max}(R)$, $\text{Min}(R)$, $\text{U}(R)$, $\text{Tot}(R)$ are respectively the set of prime ideals, maximal ideals, minimal prime ideals, units, total ring of fractions of a ring R and $\kappa(P) = R_P/PR_P$ is the residual field of R at $P \in \text{Spec}(R)$.

If $R \subseteq S$ is an extension, then $(R : S)$ is its conductor and if $P \in \text{Spec}(R)$, then S_P is the localization $S_{R \setminus P}$. We denote the integral closure of R in S by \overline{R}^S (or \overline{R}).

A local ring is here what is called elsewhere a quasi-local ring. The *support* of an R -module E is $\text{Supp}_R(E) := \{P \in \text{Spec}(R) \mid E_P \neq 0\}$ and $\text{MSupp}_R(E) := \text{Supp}_R(E) \cap \text{Max}(R)$. Finally, \subset denotes proper inclusion and $|X|$ the cardinality of a set X .

Scholium A We give some recalls about flat epimorphisms (see [27, Chapitre IV], except (2) which is [31, Proposition 2]).

- (1) $R \rightarrow S$ is a flat epimorphism \Leftrightarrow for all $P \in \text{Spec}(R)$, either $R_P \rightarrow S_P$ is an isomorphism or $S = PS \Leftrightarrow R_P \subseteq S_P$ is a flat epimorphism for all $P \in \text{Spec}(R)$.
- (2) (S) A flat epimorphism, with a zero-dimensional domain, is surjective.
- (3) If $f : A \rightarrow B$ and $g : B \rightarrow C$ are ring morphisms such that $g \circ f$ is injective and f is a flat epimorphism, then g is injective.
- (4) Let $R \subseteq T \subseteq S$ be a tower of extensions, such that $R \subseteq S$ is a flat epimorphism. Then $T \subseteq S$ is a flat epimorphism but $R \subseteq T$ does not need. A Prüfer extension remedies to this defect.

- (5) (L) A faithfully flat epimorphism is an isomorphism. Hence, $R = S$ if $R \subseteq S$ is an integral flat epimorphism.
- (6) If $f : R \rightarrow S$ is a flat epimorphism and J an ideal of S , then $J = f^{-1}(J)S$.
- (7) If $f : R \rightarrow S$ is an epimorphism, then f is spectrally injective and its residual extensions are isomorphisms.
- (8) Flat epimorphisms remain flat epimorphisms under base change (in particular, after a localization with respect to a multiplicatively closed subset).
- (9) Flat epimorphisms are descended by faithfully flat morphisms.

1.2. Recalls and results on Prüfer extensions. We recall some definitions and properties of ring extensions $R \subseteq S$ and rings R . There are a lot of characterizations of Prüfer extensions. We keep only those that are useful in this paper. We give the two definitions that are dual and emphasize some characterizations in the local case.

Scholium B

- (1) [26] $R \subseteq S$ is called Prüfer if $R \subseteq T$ is a flat epimorphism for each $T \in [R, S]$.
- (2) $R \subseteq S$ is called a *normal* pair if $T \subseteq S$ is integrally closed for each $T \in [R, S]$.
- (3) $R \subseteq S$ is Prüfer if and only if it is a normal pair [26, Theorem 5.2(4)].
- (4) R is called Prüfer if its finitely generated regular ideals are invertible, or equivalently, $R \subseteq \text{Tot}(R)$ is Prüfer [20, Theorem 13((5)(9))].

Hence Prüfer extensions are a relativization of Prüfer rings. Clearly, a minimal extension is a flat epimorphism if and only if it is Prüfer. We will then use for such extensions the terminology: *Prüfer minimal* extensions. The reader may find some properties of Prüfer minimal extensions in [36, Proposition 3.2, Lemma 3.4 and Proposition 3.5], asserted by L. Dechene in her dissertation, but where in addition R must be supposed local. The reason why is that this word has surprisingly disappeared during the printing process of [36].

We will need the two next results. Some of them do not explicitly appear in [26] but deserve to be emphasized. We refer to [26, Definition 1, p.22] for a definition of Manis extensions.

Proposition 1.1. *Let $R \subseteq S$ be a ring extension.*

- (1) $R \subseteq S$ is Prüfer if and only if $R_P \subseteq S_P$ is Prüfer for each $P \in \text{Spec}(R)$ (respectively, $P \in \text{Supp}(S/R)$).

- (2) $R \subseteq S$ is Prüfer if and only if $R_M \subseteq S_M$ is Manis for each $M \in \text{Max}(R)$.

Proof. (1) The class of Prüfer extensions is stable under localization [26, Proposition 5.1(ii), p.46-47]. To get the converse, use Scholium A(1). (2) follows from [26, Proposition 2.10, p.28, Definition 1, p.46]. \square

Proposition 1.2. *Let $R \subseteq S$ be a ring extension, where R is local.*

- (1) $R \subseteq S$ is Manis if and only if $S \setminus R \subseteq U(S)$ and $x \in S \setminus R \Rightarrow x^{-1} \in R$. In that case, $R \subseteq S$ is integrally closed.
- (2) $R \subseteq S$ is Manis if and only if $R \subseteq S$ is Prüfer.
- (3) $R \subseteq S$ is Prüfer if and only if there exists $P \in \text{Spec}(R)$ such that $S = R_P$, $P = SP$ and R/P is a valuation domain. Under these conditions, S/P is the quotient field of R/P .

Proof. (1) is [26, Theorem 2.5, p.24]. (2) is [26, Scholium 10.4, p.147]. Then (3) is [10, Theorem 6.8]. \square

Next result shows that Prüfer FCP extensions can be described in a special manner.

Proposition 1.3. *Let $R \subset S$ be a ring extension.*

- (1) If $R \subset S$ has FCP, then $R \subset S$ is integrally closed $\Leftrightarrow R \subset S$ is Prüfer $\Leftrightarrow R \subset S$ is a composite of Prüfer minimal extensions.
- (2) If $R \subset S$ is integrally closed, then $R \subset S$ has FCP $\Leftrightarrow R \subset S$ is Prüfer and $\text{Supp}(S/R)$ is finite.

Proof. (1) Assume that $R \subset S$ has FCP. If $R \subset S$ is integrally closed, then, $R \subset S$ is composed of Prüfer minimal extensions by [10, Lemma 3.10]. Conversely, if $R \subset S$ is composed of Prüfer minimal extensions, $R \subset S$ is integrally closed, since so is each Prüfer minimal extension. A Prüfer extension is obviously integrally closed, and an FCP integrally closed extension is Prüfer by [10, Theorem 6.3].

- (2) The logical equivalence is [10, Theorem 6.3]. \square

Definition 1.4. [26] A ring extension $R \subseteq S$ has:

- (1) a greatest flat epimorphic subextension $R \subseteq \widehat{R}^S$, called the **Morita hull** of R in S .
- (2) a greatest Prüfer subextension $R \subseteq \widetilde{R}^S$, called the **Prüfer hull** of R in S .

We set $\widehat{R} := \widehat{R}^S$ and $\widetilde{R} := \widetilde{R}^S$, if no confusion can occur. $R \subseteq S$ is called Prüfer-closed if $R = \widetilde{R}$.

Note that \widetilde{R}^S is denoted by $P(R, S)$ in [26] and \widehat{R}^S is the weakly surjective hull $M(R, S)$ of [26]. Our terminology is justified because

Morita's work is earlier [30, Corollary 3.4]. The Morita hull can be computed by using a (transfinite) induction [30]. Let S' be the set of all $s \in S$ such that there is some ideal I of R , such that $IS = S$ and $Is \subseteq R$. Then $R \subseteq S'$ is a subextension of $R \subseteq S$. We set $S_1 := S'$ and $S_{i+1} := (S_i)' \subseteq S_i$. By [30, p.36], if $R \subseteq S$ is an FCP extension, then $\widehat{R} = S_n$ for some integer n .

At this stage it is interesting to point out a result; showing again that integral closedness and Prüfer extensions are closely related.

Proposition 1.5. *Olivier [33, Corollary, p.56] An extension $R \subseteq S$ is integrally closed if and only if there is a pullback square:*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ V & \longrightarrow & K \end{array}$$

where V is a semi-hereditary ring and K its total quotient ring.

In that case $V \subseteq K$ is a Prüfer extension, since V is a Prüfer ring, whose localizations at prime ideals are valuation domains and K is an absolutely flat ring. As there exist integrally closed extensions that are not Prüfer, we see in passing that the pullback construction may not descend Prüfer extensions. The above result has a companion for minimal extensions that are Prüfer [21, Proposition 3.2].

Proposition 1.6. *Let $R \subseteq S$ be an extension and $T \in [R, S]$, then $\widetilde{R}^T = \widetilde{R} \cap T$. Therefore, for $T, U \in [R, S]$ with $T \subseteq U$, then $\widetilde{R}^T \subseteq \widetilde{R}^U$.*

Proof. Obvious, since the Prüfer hull \widetilde{R}^T is the greatest Prüfer extension $R \subseteq V$ contained in T . \square

We will show later that in some cases $\widetilde{T} \subseteq \widetilde{U}$ if $R \subseteq S$ has FCP.

2. QUASI-PRÜFER EXTENSIONS

We introduced the following definition in [38, p.10].

Definition 2.1. An extension of rings $R \subseteq S$ is called quasi-Prüfer if one of the following equivalent statements holds:

- (1) $\overline{R} \subseteq S$ is a Prüfer extension;
- (2) $R \subseteq S$ can be factored $R \subseteq T \subseteq S$, where $R \subseteq T$ is integral and $T \subseteq S$ is Prüfer. In that case $\overline{R} = T$

To see that (2) \Rightarrow (1) observe that if (2) holds, then $T \subseteq \overline{R}$ is integral and a flat injective epimorphism, so that $\overline{R} = T$ by (L) (Scholium A(5)).

We observe that quasi-Prüfer extensions are akin to quasi-finite extensions if we refer to Zariski Main Theorem. This will be explored in Section 6, see for example Theorem 6.2.

Hence integral or Prüfer extensions are quasi-Prüfer. An extension is clearly Prüfer if and only if it is quasi-Prüfer and integrally closed. Quasi-Prüfer extensions allow us to avoid FCP hypotheses.

We give some other definitions involved in ring extensions $R \subseteq S$. The *fiber* at $P \in \text{Spec}(R)$ of $R \subseteq S$ is $\text{Fib}_{R,S}(P) := \{Q \in \text{Spec}(S) \mid Q \cap R = P\}$. The subspace $\text{Fib}_{R,S}(P)$ of $\text{Spec}(S)$ is homeomorphic to the spectrum of the fiber ring $F_{R,S}(P) := \kappa(P) \otimes_R S$ at P . The homeomorphism is given by the spectral map of $S \rightarrow \kappa(P) \otimes_R S$ and $\kappa(P) \rightarrow \kappa(P) \otimes_R S$ is the *fiber morphism* at P .

Definition 2.2. A ring extension $R \subseteq S$ is called:

- (1) *incomparable* if for each pair $Q \subseteq Q'$ of prime ideals of S , then $Q \cap R = Q' \cap R \Rightarrow Q = Q'$, or equivalently, $\kappa(P) \otimes_R T$ is a zero-dimensional ring for each $T \in [R, S]$ and $P \in \text{Spec}(R)$, such that $\kappa(P) \otimes_R T \neq 0$.
- (2) an *INC-pair* if $R \subseteq T$ is incomparable for each $T \in [R, S]$.
- (3) *residually algebraic* if $R/(Q \cap R) \subseteq S/Q$ is algebraic for each $Q \in \text{Spec}(S)$.
- (4) a *residually algebraic pair* if the extension $R \subseteq T$ is residually algebraic for each $T \in [R, S]$.

The following characterization was announced in [38]. We were unaware that this result is also proved in [7, Corollary 1], when we present it in ArXiv. However, our proof is largely shorter because we use the powerful results of [26].

Theorem 2.3. *An extension $R \subseteq S$ is quasi-Prüfer if and only if $R \subseteq S$ is an INC-pair and, if and only if, $R \subseteq S$ is a residually algebraic pair.*

Proof. Suppose that $R \subseteq S$ is quasi-Prüfer and let $T \in [R, S]$. We set $U := \overline{R}T$. Then $\overline{R} \subseteq U$ is a flat epimorphism by definition of a Prüfer extension and hence is incomparable as is $R \subseteq \overline{R}$. It follows that $R \subseteq U$ is incomparable. Since $T \subseteq U$ is integral, it has going-up. It follows that $R \subseteq T$ is incomparable. Conversely, if $R \subseteq S$ is an INC-pair, then so is $\overline{R} \subseteq S$. Since $\overline{R} \subseteq S$ is integrally closed, $\overline{R} \subseteq S$ is Prüfer [26, Theorem 5.2,(9'), p.48]. The second equivalence is [14, Proposition 2.1] or [18, Theorem 6.5.6]. \square

Corollary 2.4. *An extension $R \subseteq S$ is quasi-Prüfer if and only if $\overline{R} \subseteq \overline{T}$ is Prüfer for each $T \in [R, S]$.*

It follows that most of the properties described in [6] for integrally closed INC-pairs of domains are valid for arbitrary ring extensions. Moreover, a result of Dobbs is easily gotten: an INC-pair $R \subseteq S$ is an integral extension if and only if $\overline{R} \subseteq S$ is spectrally surjective [14, Theorem 2.2]. This follows from Scholium A, Property (L).

Example 2.5. Quasi-Prüfer domains R with quotient fields K can be characterized by $R \subseteq K$ is quasi-Prüfer. The reader may consult [9, Theorem 1.1] or [18]. In view of [2, Theorem 2.7], R is a quasi-Prüfer domain if and only if $\text{Spec}(R(X)) \rightarrow \text{Spec}(R)$ is bijective.

We give here another example of quasi-Prüfer extension. An extension $R \subseteq S$ is called a *going-down pair* if each of its subextensions has the going-down property. For such a pair, $R \subseteq T$ has incomparability for each $T \in [R, S]$, at each non-maximal prime ideal of R [3, Lemma 5.8](ii). Now let M be a maximal ideal of R , whose fiber is not void in T . Then $R \subseteq T$ is a going-down pair, and so is $R/M \subseteq T/MT$ because $MT \cap R = M$. By [3, Corollary 5.6], the dimension of T/MT is ≤ 1 . Therefore, if $R \subseteq S$ is a going-down pair, then $R \subseteq S$ is quasi-Prüfer if and only if $\dim(T/MT) \neq 1$ for each $T \in [R, S]$ and $M \in \text{Max}(R)$.

Also *open-ring pairs* $R \subseteq S$ are quasi-Prüfer by [8, Proposition 2.13].

An *i-pair* is an extension $R \subseteq S$ such that $\text{Spec}(T) \rightarrow \text{Spec}(R)$ is injective for each $T \in [R, S]$, or equivalently if and only if $R \subseteq S$ is quasi-Prüfer and $R \subseteq \overline{R}$ is spectrally injective [38, Proposition 5.8]. These extensions appear frequently in the integral domains context. Another examples are given by some extensions $R \subseteq S$, such that $\text{Spec}(S) = \text{Spec}(R)$ as sets, as we will see later.

3. PROPERTIES OF QUASI-PRÜFER EXTENSIONS

We now develop the machinery of quasi-Prüfer extensions.

Proposition 3.1. *An extension $R \subseteq S$ is (quasi-)Prüfer if and only if $R_P \subseteq S_P$ is (quasi-)Prüfer for any $P \in \text{Spec}(R)$ ($P \in \text{MSupp}(S/R)$).*

Proof. The proof is easy if we use the INC-pair property definition of quasi-Prüfer extension (see also [6, Proposition 2.4]). \square

Proposition 3.2. *Let $R \subseteq S$ be a quasi-Prüfer extension and $\varphi : S \rightarrow S'$ an integral ring morphism. Then $\varphi(R) \subseteq S'$ is quasi-Prüfer and $S' = \varphi(S)\overline{\varphi(R)}$, where $\overline{\varphi(R)}$ is the integral closure of $\varphi(R)$ in S' .*

Proof. It is enough to apply [26, Theorem 5.9] to the Prüfer extension $\overline{R} \subseteq S$ and to use Definition 2.1. \square

This result applies with $S' := S \otimes_R R'$, where $R \rightarrow R'$ is an integral morphism. Therefore integrality ascends the quasi-Prüfer property.

We know that a composite of Prüfer extensions is Prüfer [26, Theorem 5.6, p.51]. The following Corollary 3.3 contains [7, Theorem 3].

Corollary 3.3. *Let $R \subseteq T \subseteq S$ be a tower of extensions. Then $R \subseteq S$ is quasi-Prüfer if and only if $R \subseteq T$ and $T \subseteq S$ are quasi-Prüfer. It follows that $R \subseteq T$ is quasi-Prüfer if and only if $R \subseteq \overline{RT}$ is quasi-Prüfer.*

Proof. Consider a tower (\mathcal{T}) of extensions $R \subseteq \overline{R} \subseteq S := R' \subseteq \overline{R'} \subseteq S'$ (a composite of two quasi-Prüfer extensions). By using Proposition 3.2 we see that $\overline{R} \subseteq S = R' \subseteq \overline{R'}$ is quasi-Prüfer. Then (\mathcal{T}) is obtained by writing on the left an integral extension and on the right a Prüfer extension. Therefore, (\mathcal{T}) is quasi-Prüfer. We prove the converse.

If $R \subseteq T \subseteq S$ is a tower of extensions, then $R \subseteq T$ and $T \subseteq S$ are INC-pairs whenever $R \subseteq S$ is an INC-pair. The converse is then a consequence of Theorem 2.3.

The last statement is [7, Corollary 4]. \square

Using the above corollary, we can exhibit new examples of quasi-Prüfer extensions. We recall that a ring R is called *Laskerian* if each of its ideals is a finite intersection of primary ideals and a ring extension $R \subset S$ a *Laskerian pair* if each $T \in [R, S]$ is a Laskerian ring. Then [42, Proposition 2.1] shows that if R is an integral domain with quotient field $F \neq R$ and $F \subset K$ is a field extension, then $R \subset K$ is a Laskerian pair if and only if K is algebraic over R and \overline{R} (in K) is a Laskerian Prüfer domain. It follows easily that $R \subset K$ is quasi-Prüfer.

Next result generalizes [25, Proposition 1].

Corollary 3.4. *An FMC extension $R \subset S$ is quasi-Prüfer.*

Proof. Because $R \subset S$ is a composite of finitely many minimal extensions, by Corollary 3.3, it is enough to observe that a minimal extension is either Prüfer or integral. \square

Corollary 3.5. *Let $R \subseteq S$ be a quasi-Prüfer extension and a tower $R \subseteq T \subseteq S$, where $R \subseteq T$ is integrally closed. Then $R \subseteq T$ is Prüfer.*

Proof. Observe that $R \subseteq T$ is quasi-Prüfer and then that $R = \overline{R}^T$. \square

Next result deals with the Dobbs-Shapiro *pseudo-Prüfer* extensions of integral domains [15], that they called pseudo-normal pairs. Suppose that R is local, we call here pseudo-Prüfer an extension $R \subseteq S$ such that there exists $T \in [R, S]$ with $\text{Spec}(R) = \text{Spec}(T)$ and $T \subseteq S$ is Prüfer [15, Corollary 2.5]. If R is arbitrary, the extension $R \subseteq S$ is called pseudo-Prüfer if $R_M \subseteq S_M$ is pseudo-Prüfer for each $M \in \text{Max}(R)$. In view of the Corollary 3.3, it is enough to characterize quasi-Prüfer extensions of the type $R \subseteq T$ with $\text{Spec}(R) = \text{Spec}(T)$.

Corollary 3.6. *Let $R \subseteq T$ be an extension with $\text{Spec}(R) = \text{Spec}(T)$ and (R, M) local. Then $R \subseteq T$ is quasi-Prüfer if and only if $\text{Spec}(R) = \text{Spec}(U)$ for each $U \in [R, T]$ and, if and only if $R/M \subseteq T/M$ is an algebraic field extension. In such a case, $R \subseteq T$ is Prüfer-closed.*

Proof. It follows from [1] that $M \in \text{Max}(T)$. Part of the proof is gotten by observing that $R \subseteq U$ is an INC extension if $\text{Spec}(R) = \text{Spec}(U)$. Another one is proved in [1, Corollary 3.26]. Now $R \subseteq \tilde{R}$ is a spectrally surjective flat epimorphism and then, by Scholium A, $R = \tilde{R}$. \square

Let $R \subseteq S$ be an extension and I an ideal shared with R and S . It is easy to show that $R \subseteq S$ is quasi-Prüfer if and only if $R/I \subseteq S/I$ is quasi-Prüfer by using [26, Proposition 5.8] in the Prüfer case. We are able to give a more general statement.

Lemma 3.7. *Let $R \subseteq S$ be a (quasi-)Prüfer extension and J an ideal of S with $I = J \cap R$. Then $R/I \subseteq S/J$ is a (quasi-)Prüfer extension. If $R \subseteq S$ is Prüfer and N is a maximal ideal of S , then $R/(N \cap R)$ is a valuation domain with quotient field S/N .*

Proof. Assume first that $R \subseteq S$ is Prüfer. We have $J = IS$ by Scholium A(6), because $R \subseteq S$ is a flat epimorphism. Therefore, any $D \in [R/I, S/J]$ is of the form C/J where $C \in [R, S]$. We can write $C/IS = (C + I)/IS \cong C/C \cap IS$. As $R \subseteq C$ is a flat epimorphism, $C \cap IS = IC$. It then follows that $D = C \otimes R/I$ and we get easily that $R/I \subseteq S/J$ is Prüfer, since $R/I \subseteq D$ is a flat epimorphism. The quasi-Prüfer case is an easy consequence. \square

With this lemma we generalize and complete [23, Proposition 1.1].

Proposition 3.8. *Let $R \subseteq S$ be an extension of rings. The following statements are equivalent:*

- (1) $R \subseteq S$ is quasi-Prüfer;
- (2) $R/(Q \cap R) \subseteq S/Q$ is quasi-Prüfer for each $Q \in \text{Spec}(S)$;
- (3) $(X - s)S[X] \cap R[X] \not\subseteq M[X]$ for each $s \in S$ and $M \in \text{Max}(R)$;
- (4) For each $T \in [R, S]$, the fiber morphisms of $R \subseteq T$ are integral.

Proof. (1) \Rightarrow (2) is entailed by Lemma 3.7. Assume that (2) holds and let $M \in \text{Max}(R)$ that contains a minimal prime ideal P , lain over by a minimal prime ideal Q of S . Then (2) \Rightarrow (3) follows from [23, Proposition 1.1(1)], applied to $R/(Q \cap R) \subseteq S/Q$. If (3) holds, argue as in the paragraph before [23, Proposition 1.1] to get that $R \subseteq S$ is a \mathcal{P} -extension, whence an INC-extension by [14, Proposition 2.1]. Because integral extensions have incomparability, we see that (4) \Rightarrow (1). Corollary 3.3 shows that the reverse implication holds, if any quasi-Prüfer

extension $R \subseteq S$ has integral fiber morphisms. For $P \in \text{Spec}(R)$, the extension $R_P/PR_P \subseteq S_P/PS_P$ is quasi-Prüfer by Lemma 3.7. The ring $\overline{R}_P/P\overline{R}_P$ is zero-dimensional and $\overline{R}_P/P\overline{R}_P \rightarrow S_P/PS_P$, being a flat epimorphism, is therefore surjective by Scholium A (S). It follows that the fiber morphism at P is integral. \square

Remark 3.9. The logical equivalence (1) \Leftrightarrow (2) is still valid if we replace quasi-Prüfer with integral in the above proposition. It is enough to show that an extension $R \subseteq S$ is integral when $R/P \subseteq S/Q$ is integral for each $Q \in \text{Spec}(S)$ and $P := Q \cap R$. We can suppose that $S = R[s] \cong R[X]/I$, where X is an indeterminate, I an ideal of $R[X]$ and Q varies in $\text{Min}(S)$, because for an extension $A \subseteq B$, any element of $\text{Min}(A)$ is lain over by some element of $\text{Min}(B)$. If Σ is the set of unitary polynomials of $R[X]$, the assumptions show that any element of $\text{Spec}(R[X])$, containing I , meets Σ . As Σ is a multiplicatively closed subset, $I \cap \Sigma \neq \emptyset$, whence s is integral over R .

But a similar result does not hold if we replace quasi-Prüfer with Prüfer, except if we suppose that $R \subseteq S$ is integrally closed. To see this, apply the above proposition to get a quasi-Prüfer extension $R \subseteq S$ if each $R/P \subseteq S/Q$ is Prüfer. Actually, this situation already occurs for Prüfer rings and their factor domains, as Lucas's paper [29] shows. More precisely, [29, Proposition 2.7] and the third paragraph of [29, p. 336] shows that if R is a ring with $\text{Tot}(R)$ absolutely flat, then R is a quasi-Prüfer ring if R/P is a Prüfer domain for each $P \in \text{Spec}(R)$. Now example [29, Example 2.4] shows that R is not necessarily Prüfer.

We observe that if $R \subseteq S$ is quasi-Prüfer, then R/M is a quasi-Prüfer domain for each $N \in \text{Max}(S)$ and $M := N \cap R$ (in case $R \subseteq S$ is integral, R/M is a field). To prove this, observe that $R/M \subseteq S/N$ can be factored $R/M \subseteq \kappa(M) \subseteq S/N$. As we will see, $R/M \subseteq \kappa(M)$ is quasi-Prüfer because $R/M \subseteq S/N$ is quasi-Prüfer.

The class of Prüfer extensions is not stable by (flat) base change. For example, let V be a valuation domain with quotient field K . Then $V[X] \subseteq K[X]$ is not Prüfer [26, Example 5.12, p.53]. Thus if we consider an ideal I of R and $J := IS$, $R \subseteq S$ Prüfer may not imply $R/I \subseteq S/IS$ Prüfer except if $IS \cap R = I$. This happens for instance for a prime ideal I of R that is lain over by a prime ideal of S .

Proposition 3.10. *Let $R \subseteq S$ be a (quasi)-Prüfer extension and $R \rightarrow T$ a flat epimorphism, then $T \subseteq S \otimes_R T$ is (quasi)-Prüfer. If in addition S and T are both subrings of some ring and $R \subseteq T$ is an extension, then $T \subseteq TS$ is (quasi)-Prüfer.*

Proof. For the first part, it is enough to consider the Prüfer case. It is well known that the following diagram is a pushout if $Q \in \text{Spec}(T)$ is lying over P in R :

$$\begin{array}{ccc} R_P & \longrightarrow & S_P \\ \downarrow & & \downarrow \\ T_Q & \longrightarrow & (T \otimes_R S)_Q \end{array}$$

As $R_P \rightarrow T_Q$ is an isomorphism since $R \rightarrow T$ is a flat epimorphism by Scholium A, it follows that $R_P \subseteq S_P$ identifies to $T_Q \rightarrow (T \otimes_R S)_Q$. The result follows because Prüfer extensions localize and globalize.

In case $R \rightarrow T$ is a flat epimorphic extension, the surjective maps $T \otimes_R S \rightarrow TS$ and $\overline{R} \otimes_R T \rightarrow \overline{R}T$ are isomorphisms because $\overline{R} \rightarrow T\overline{R}$ (resp. $S \rightarrow ST$) is injective and $\overline{R} \rightarrow T \otimes_T \overline{R}$ (resp. $S \rightarrow S \otimes_R T$) is a flat epimorphism. Then it is enough to use Scholium A. \square

The reader may find in [26, Corollary 5.11, p.53] that if $R \subseteq A \subseteq S$ and $R \subseteq B \subseteq S$ are extensions and $R \subseteq A$ and $R \subseteq B$ are both Prüfer, then $R \subseteq AB$ is Prüfer.

Proposition 3.11. *Let $R \subseteq A$ and $R \subseteq B$ be two extensions, where A and B are subrings of a ring S . If they are both quasi-Prüfer, then $R \subseteq AB$ is quasi-Prüfer.*

Proof. Let U and V be the integral closures of R in A and B . Then $R \subseteq A \subseteq AV$ is quasi-Prüfer because $A \subseteq AV$ is integral and Corollary 3.3 applies. Using again Corollary 3.3 with $R \subseteq V \subseteq AV$, we find that $V \subseteq AV$ is quasi-Prüfer. Now Proposition 3.10 entails that $B \subseteq AB$ is quasi-Prüfer because $V \subseteq B$ is a flat epimorphism. Finally $R \subseteq AB$ is quasi-Prüfer, since a composite of quasi-Prüfer extensions. \square

It is known that an arbitrary product of extensions is Prüfer if and only if each of its components is Prüfer [26, Proposition 5.20, p.56]. The following result is an easy consequence.

Proposition 3.12. *Let $\{R_i \subseteq S_i | i = 1, \dots, n\}$ be a finite family of quasi-Prüfer extensions, then $R_1 \times \dots \times R_n \subseteq S_1 \times \dots \times S_n$ is quasi-Prüfer. In particular, if $\{R \subseteq S_i | i = 1, \dots, n\}$ is a finite family of quasi-Prüfer extensions, then $R \subseteq S_1 \times \dots \times S_n$ is quasi-Prüfer.*

In the same way we have the following result deduced from [26, Remark 5.14, p.54].

Proposition 3.13. *Let $R \subseteq S$ be an extension of rings and an upward directed family $\{R_\alpha | \alpha \in I\}$ of elements of $[R, S]$ such that $R \subseteq R_\alpha$ is quasi-Prüfer for each $\alpha \in I$. Then $R \subseteq \cup[R_\alpha | \alpha \in I]$ is quasi-Prüfer.*

Proof. It is enough to use [26, Proposition 5.13, p.54] where A_α is the integral closure of R in R_α . \square

A ring morphism $R \rightarrow T$ preserves the integral closure of ring morphisms $R \rightarrow S$ if $\overline{T}^{T \otimes_R S} \cong T \otimes_R \overline{R}$ for every ring morphism $R \rightarrow S$. An absolutely flat morphism $R \rightarrow T$ ($R \rightarrow T$ and $T \otimes_R T \rightarrow T$ are both flat) preserves integral closure [33, Theorem 5.1]. Flat epimorphisms, Henselizations and étale morphisms are absolutely flat. Another examples are morphisms $R \rightarrow T$ that are essentially of finite type and (absolutely) reduced [37, Proposition 5.19](2). Such morphisms are flat if R is reduced [28, Proposition 3.2].

We will prove an ascent result for absolutely flat ring morphisms. This will be proved by using base changes. For this we need to introduce some concepts. A ring A is called an AIC ring if each monic polynomial of $A[X]$ has a zero in A . We recalled in [35, p.4662] that any ring A has a faithfully flat integral extension $A \rightarrow A^*$, where A^* is an AIC ring. Moreover, if A is an AIC ring, each localization A_P at a prime ideal P of A is a strict Henselian ring [35, Lemma II.2].

Theorem 3.14. *Let $R \subseteq S$ be a (quasi-) Prüfer extension and $R \rightarrow T$ an absolutely flat ring morphism. Then $T \rightarrow T \otimes_R S$ is a (quasi-) Prüfer extension.*

Proof. We can suppose that R is an AIC ring. To see this, it is enough to use the base change $R \rightarrow R^*$. We set $T^* := T \otimes_R R^*$, $S^* := S \otimes_R R^*$. We first observe that $R^* \subseteq S^*$ is quasi-Prüfer for the following reason: the composite extension $R \subseteq S \subseteq S^*$ is quasi-Prüfer because the last extension is integral. Moreover, $R^* \rightarrow T^*$ is absolutely flat. In case $T^* \subseteq T^* \otimes_{R^*} S^*$ is quasi-Prüfer, so is $T \subseteq T \otimes_R S$, because $T \rightarrow T^* = T \otimes_R R^*$ is faithfully flat and $T^* \subseteq T^* \otimes_{R^*} S^*$ is deduced from $T \subseteq_R S$ by the faithfully flat base change $T \rightarrow T \otimes_R S$. It is then enough to apply Proposition 3.17.

We thus assume from now on that R is an AIC ring.

Let $N \in \text{Spec}(T)$ be lying over M in R . Then $R_M \rightarrow T_N$ is absolutely flat [32, Proposition f] and $R_M \subseteq S_M$ is quasi-Prüfer. Now observe that $(T \otimes_R S)_N \cong T_N \otimes_{R_M} S_M$. Therefore, we can suppose that R and T are local and $R \rightarrow T$ is local and injective. We deduce from [33, Theorem 5.2], that $R_M \rightarrow T_N$ is an isomorphism. Therefore the proof is complete in the quasi-Prüfer case. For the Prüfer case, we need only to observe that absolutely flat morphisms preserve integral closure and a quasi-Prüfer extension is Prüfer if it is integrally closed. \square

Proposition 3.15. *Let $R \subseteq S$ be an extension of rings and $R \rightarrow T$ a base change which preserves integral closure. If $T \subseteq T \otimes_R S$ has FCP and $R \subseteq S$ is Prüfer, then $T \subseteq T \otimes_R S$ is Prüfer.*

Proof. The result holds because an FCP extension is Prüfer if and only if it is integrally closed. \square

We observe that $T \otimes_R \tilde{R} \subseteq \tilde{T}$ needs not to be an isomorphism, since this property may fail even for a localization $R \rightarrow R_P$, where P is a prime ideal of R .

Proposition 3.16. *Let $R \subseteq S$ be an extension of rings, $R \rightarrow R'$ a faithfully flat ring morphism and set $S' := R' \otimes_R S$. If $R' \subseteq S'$ is (quasi-) Prüfer (respectively, FCP), then so is $R \subseteq S$.*

Proof. The Prüfer case is clear, because faithfully flat morphisms descend flat epimorphisms (Scholium A (9)). For the quasi-Prüfer case, we use the INC-pair characterization and the fact that $F_{R,S}(P) \rightarrow F_{R',S'}(P')$ is faithfully flat for $P' \in \text{Spec}(R')$ lying over P in R [22, Corollaire 3.4.9]. The FCP case is proved in [11, Theorem 2.2]. \square

Proposition 3.17. *Let $R \subseteq S$ be a ring extension and $R \rightarrow R'$ a spectrally surjective ring morphism (for example, either faithfully flat or injective and integral). Then $R \subseteq S$ is quasi-Prüfer if $R' \rightarrow R' \otimes_R S$ is injective (for example, if $R \rightarrow R'$ is faithfully flat) and quasi-Prüfer.*

Proof. Let $T \in [R, S]$ and $P \in \text{Spec}(R)$ and set $T' := T \otimes_R R'$. There is some $P' \in \text{Spec}(R')$ lying over P , because $R \rightarrow R'$ is spectrally surjective. There is a faithfully flat morphism $F_{R,T}(P) \rightarrow F_{R',T'}(P') \cong F_{R,T}(P) \otimes_{\mathbf{k}(P)} \kappa(P')$ [22, Corollaire 3.4.9]. By Theorem 2.3, the result follows from the faithful flatness of $F_{R,T}(P) \rightarrow F_{R',T \otimes_R R'}(P')$. \square

Theorem 3.18. *Let $R \subseteq S$ be a ring extension.*

- (1) $R \subseteq S$ has a greatest quasi-Prüfer subextension $R \subseteq \overrightarrow{\overline{R}} = \widetilde{\overline{R}}$.
- (2) $R \subseteq \overline{R}\tilde{R} =: \vec{R}$ is quasi-Prüfer and then $\vec{R} \subseteq \overrightarrow{\overline{R}}$.
- (3) $\overrightarrow{\overline{R}}^{\vec{R}} = \overline{R}$ and $\widetilde{\overline{R}}^{\vec{R}} = \tilde{R}$.

Proof. To see (1), use Proposition 3.13 which tells us that the set of all quasi-Prüfer subextensions is upward directed and then use Proposition 3.12 to prove the existence of $\overrightarrow{\overline{R}}$. Then let $R \subseteq T \subseteq \overrightarrow{\overline{R}}$ be a tower with $R \subseteq T$ integral and $T \subseteq \overrightarrow{\overline{R}}$ Prüfer. From $T \subseteq \overline{R} \subseteq \widetilde{\overline{R}} \subseteq \overrightarrow{\overline{R}}$, we deduce that $T = \overline{R}$ and then $\overrightarrow{\overline{R}} = \widetilde{\overline{R}}$.

(2) Now $R \subseteq \overline{R}\tilde{R}$ can be factored $R \subseteq \tilde{R} \subseteq \overline{R}\tilde{R}$ and is a tower of quasi-Prüfer extensions, because $\tilde{R} \rightarrow \overline{R}\tilde{R}$ is integral.

(3) Clearly, the integral closure and the Prüfer closure of R in \overrightarrow{R} are the respective intersections of \overline{R} and \widetilde{R} with \overrightarrow{R} , and $\overline{R}, \widetilde{R} \subseteq \overrightarrow{R}$. \square

This last result means that, as long integral closures and Prüfer closures of subsets of \overrightarrow{R} are concerned, we can suppose that $R \subseteq S$ is quasi-Prüfer.

4. ALMOST-PRÜFER EXTENSIONS

We next give a definition “dual” of the definition of a quasi-Prüfer extension.

4.1. Arbitrary extensions.

Definition 4.1. A ring extension $R \subseteq S$ is called an *almost-Prüfer* extension if it can be factored $R \subseteq T \subseteq S$, where $R \subseteq T$ is Prüfer and $T \subseteq S$ is integral.

Proposition 4.2. *An extension $R \subseteq S$ is almost-Prüfer if and only if $\widetilde{R} \subseteq S$ is integral. It follows that the subring T of the above definition is $\widetilde{R} = \widehat{R}$ when $R \subseteq S$ is almost-Prüfer.*

Proof. If $R \subseteq S$ is almost-Prüfer, there is a factorization $R \subseteq T \subseteq \widetilde{R} \subseteq \widehat{R} \subseteq S$, where $T \subseteq \widehat{R}$ is both integral and a flat epimorphism by Scholium A (4). Therefore, $T = \widetilde{R} = \widehat{R}$ by Scholium A (5) (L). \square

Corollary 4.3. *Let $R \subseteq S$ be a quasi-Prüfer extension, and let $T \in [R, S]$. Then, $T \cap \overline{R} \subseteq T \overline{R}$ is almost-Prüfer. Moreover, $T = \widetilde{\overline{R} \cap T}^{\overline{R}}$.*

Proof. $T \cap \overline{R} \subseteq T$ is quasi-Prüfer by Corollary 3.3. Being integrally closed, it is Prüfer by Corollary 3.5. Moreover, $T \subseteq T \overline{R}$ is an integral extension. Then, $T \cap \overline{R} \subseteq T \overline{R}$ is almost-Prüfer and $T = \widetilde{\overline{R} \cap T}^{\overline{R}}$. \square

We note that integral extensions and Prüfer extensions are almost-Prüfer and hence minimal extensions are almost-Prüfer. There are quasi-Prüfer extensions that are not almost-Prüfer. It is enough to consider [39, Example 3.5(1)]. Let $R \subseteq T \subseteq S$ be two minimal extensions, where R is local, $R \subseteq T$ integral and $T \subseteq S$ is Prüfer. Then $R \subseteq S$ is quasi-Prüfer but not almost-Prüfer, because $S = \widehat{R}$ and $R = \widetilde{R}$. The same example shows that a composite of almost-Prüfer extensions may not be almost-Prüfer.

But the reverse implication holds.

Theorem 4.4. *Let $R \subseteq S$ be an almost-Prüfer extension. Then $R \subseteq S$ is quasi-Prüfer. Moreover, $\widetilde{R} = \widehat{R}$, $(\widetilde{R})_P = \widehat{R}_P$ for each $P \in \text{Spec}(R)$. In this case, any flat epimorphic subextension $R \subseteq T$ is Prüfer.*

Proof. Let $R \subseteq \widetilde{R} \subseteq S$, be an almost-Prüfer extension, that is $\widetilde{R} \subseteq S$ is integral. The result follows because $R \subseteq \widetilde{R}$ is Prüfer. Now the Morita hull and the Prüfer hull coincide by Proposition 4.2. In the same way, $(\widetilde{R})_P \rightarrow \widehat{R}_P$ is a flat epimorphism and $(\widetilde{R})_P \rightarrow S_P$ is integral. \square

We could define almost-Prüfer rings as the rings R such that $R \subseteq \text{Tot}(R)$ is almost-Prüfer. But in that case $\widetilde{R} = \text{Tot}(R)$ (by Theorem 4.4), so that R is a Prüfer ring. The converse evidently holds. Therefore, this concept does not define something new.

We observed in [10, Remark 2.9(c)] that there is an almost-Prüfer FMC extension $R \subseteq S \subseteq T$, where $R \subseteq S$ is a Prüfer minimal extension and $S \subseteq T$ is minimal and integral. But $R \subseteq T$ is not an FCP extension.

Proposition 4.5. *Let $R \subseteq S$ be an extension verifying the hypotheses:*

- (i) $R \subseteq S$ is quasi-Prüfer.
 - (ii) $R \subseteq S$ can be factored $R \subseteq T \subseteq S$, where $R \subseteq T$ is a flat epimorphism.
- (1) *Then the following commutative diagram (D) is a pushout,*

$$\begin{array}{ccc} R & \longrightarrow & \overline{R} \\ \downarrow & & \downarrow \\ T & \longrightarrow & T\overline{R} \end{array}$$

$T\overline{R} \subseteq S$ is Prüfer and $R \subseteq T\overline{R}$ is quasi-Prüfer. Moreover, $F_{R,\overline{R}}(P) \cong F_{T,T\overline{R}}(Q)$ for each $Q \in \text{Spec}(T)$ and $P := Q \cap R$.

- (2) *If in addition $R \subseteq T$ is integrally closed, (D) is a pullback, $T \cap \overline{R} = R$, $(R : \overline{R}) = (T : T\overline{R}) \cap R$ and $(T : T\overline{R}) = (R : \overline{R})T$.*

Proof. (1) Consider the injective composite map $\overline{R} \rightarrow \overline{R} \otimes_R T \rightarrow T\overline{R}$. As $\overline{R} \rightarrow \overline{R} \otimes_R T$ is a flat epimorphism, because deduced by a base change of $R \rightarrow T$, we get that the surjective map $\overline{R} \otimes_R T \rightarrow T\overline{R}$ is an isomorphism by Scholium A (3). By fibers transitivity, we have $F_{T,\overline{R}T}(Q) \cong \kappa(Q) \otimes_{\kappa(P)} F_{R,\overline{R}}(P)$ [22, Corollaire 3.4.9]. As $\kappa(P) \rightarrow \kappa(Q)$ is an isomorphism by Scholium A, we get that $F_{R,\overline{R}}(P) \cong F_{T,\overline{R}T}(Q)$.

(2) As in [5, Lemma 3.5], $R = T \cap \overline{R}$. The first statement on the conductors has the same proof as in [5, Lemma 3.5]. The second holds because $R \subseteq T$ is a flat epimorphism (see Scholium A (6)). \square

Theorem 4.6. *Let $R \subset S$ be a quasi-Prüfer extension and the diagram (D') :*

$$\begin{array}{ccc} R & \longrightarrow & \overline{R} \\ \downarrow & & \downarrow \\ \widetilde{R} & \longrightarrow & \widetilde{R}\overline{R} \end{array}$$

- (1) (D') is a pushout and a pullback, such that $\overline{R} \cap \widetilde{R} = R$ and $(R : \overline{R}) = (\widetilde{R} : \widetilde{R}\overline{R}) \cap R$ so that $(\widetilde{R} : \widetilde{R}\overline{R}) = (R : \overline{R})\widetilde{R}$.
- (2) $R \subset S$ can be factored $R \subseteq \widetilde{R}\overline{R} = \widetilde{R} = \vec{R} \subseteq \overrightarrow{\widetilde{R}} = \widetilde{\vec{R}} = S$, where the first extension is almost-Prüfer and the second is Prüfer.
- (3) $R \subset S$ is almost-Prüfer if and only if $S = \overline{R}\widetilde{R} \Leftrightarrow \widetilde{R} = \overline{\vec{R}}$.
- (4) $R \subseteq \widetilde{R}\overline{R} = \widetilde{R} = \vec{R}$ is the greatest almost-Prüfer subextension of $R \subseteq S$ and $\widetilde{R} = \widetilde{R}^{\vec{R}}$.
- (5) $\text{Supp}(S/R) = \text{Supp}(\widetilde{R}/R) \cup \text{Supp}(\overline{R}/R)$ if $R \subseteq S$ is almost-Prüfer. (Supp can be replaced with MSupp).

Proof. To show (1), (2), in view of Theorem 3.18, it is enough to apply Proposition 4.5 with $T = \widetilde{R}$ and $S = \overrightarrow{\widetilde{R}}$, because $R \subseteq \widetilde{R}\overline{R}$ is almost-Prüfer whence quasi-Prüfer, keeping in mind that a Prüfer extension is integrally closed, whereas an integral Prüfer extension is trivial. Moreover, $\widetilde{R} = \overline{R}\widetilde{R}$ because $\overline{R}\widetilde{R} \subseteq \widetilde{R}$ is both integral and integrally closed.

(3) is obvious.

(4) Now consider an almost-Prüfer subextension $R \subseteq T \subseteq U$, where $R \subseteq T$ is Prüfer and $T \subseteq U$ is integral. Applying (3), we see that $U = \overline{R}^U \widetilde{R}^U \subseteq \overline{R}\widetilde{R}$ in view of Proposition 1.6.

(5) Obviously, $\text{Supp}(\widetilde{R}/R) \cup \text{Supp}(\overline{R}/R) \subseteq \text{Supp}(S/R)$. Conversely, let $M \in \text{Spec}(R)$ be such that $R_M \neq S_M$, and $R_M = (\widetilde{R})_M = \overline{R}_M$. Then (3) entails that $S_M = (\overline{R})_M(\widetilde{R})_M = R_M$, which is absurd. \square

Corollary 4.7. *Let $R \subseteq S$ be an almost-Prüfer extension. The following conditions are equivalent:*

- (1) $\text{Supp}(S/\overline{R}) \cap \text{Supp}(\overline{R}/R) = \emptyset$.
- (2) $\text{Supp}(S/\widetilde{R}) \cap \text{Supp}(\widetilde{R}/R) = \emptyset$.
- (3) $\text{Supp}(\widetilde{R}/R) \cap \text{Supp}(\overline{R}/R) = \emptyset$.

Proof. Since $R \subseteq S$ is almost-Prüfer, we get $(\widetilde{R})_P = \widetilde{R}_P$ for each $P \in \text{Spec}(R)$. Moreover, $\text{Supp}(S/R) = \text{Supp}(\widetilde{R}/R) \cup \text{Supp}(\overline{R}/R) = \text{Supp}(S/\overline{R}) \cup \text{Supp}(\overline{R}/R) = \text{Supp}(S/\widetilde{R}) \cup \text{Supp}(\widetilde{R}/R)$.

(1) \Rightarrow (2): Assume that there exists $P \in \text{Supp}(S/\widetilde{R}) \cap \text{Supp}(\widetilde{R}/R)$. Then, $(\widetilde{R})_P \neq S_P$, R_P , so that $R_P \subset S_P$ is neither Prüfer, nor integral.

But, $P \in \text{Supp}(S/R) = \text{Supp}(S/\overline{R}) \cup \text{Supp}(\overline{R}/R)$. If $P \in \text{Supp}(S/\overline{R})$, then $P \notin \text{Supp}(\overline{R}/R)$, so that $(\overline{R})_P = R_P$ and $R_P \subset S_P$ is Prüfer, a contradiction. If $P \in \text{Supp}(\overline{R}/R)$, then $P \notin \text{Supp}(S/\overline{R})$, so that $(\overline{R})_P = S_P$ and $R_P \subset S_P$ is integral, a contradiction.

(2) \Rightarrow (3): Assume that there exists $P \in \text{Supp}(\tilde{R}/R) \cap \text{Supp}(\overline{R}/R)$. Then, $R_P \neq (\tilde{R})_P, (\overline{R})_P$, so that $R_P \subset S_P$ is neither Prüfer, nor integral. But, $P \in \text{Supp}(S/R) = \text{Supp}(S/\tilde{R}) \cup \text{Supp}(\tilde{R}/R)$. If $P \in \text{Supp}(S/\tilde{R})$, then $P \notin \text{Supp}(\tilde{R}/R)$, so that $(\tilde{R})_P = R_P$ and $R_P \subset S_P$ is integral, a contradiction. If $P \in \text{Supp}(\tilde{R}/R)$, then $P \notin \text{Supp}(S/\tilde{R})$, so that $(\tilde{R})_P = S_P$ and $R_P \subset S_P$ is Prüfer, a contradiction.

(3) \Rightarrow (1): Assume that there exists $P \in \text{Supp}(S/\overline{R}) \cap \text{Supp}(\overline{R}/R)$. Then, $(\overline{R})_P \neq R_P, S_P$, so that $R_P \subset S_P$ is neither Prüfer, nor integral. But, $P \in \text{Supp}(S/R) = \text{Supp}(\overline{R}/R) \cup \text{Supp}(\tilde{R}/R)$. If $P \in \text{Supp}(\tilde{R}/R)$, then $P \notin \text{Supp}(\overline{R}/R)$, so that $(\overline{R})_P = R_P$ and $R_P \subset S_P$ is Prüfer, a contradiction. If $P \in \text{Supp}(\overline{R}/R)$, then $P \notin \text{Supp}(\tilde{R}/R)$, so that $(\tilde{R})_P = R_P$ and $R_P \subset S_P$ is integral, a contradiction. \square

Proposition 4.5 has the following similar statement proved by Ayache and Dobbs. It reduces to Theorem 4.6 in case $R \subseteq S$ has FCP because of Proposition 1.3.

Proposition 4.8. *Let $R \subseteq T \subseteq S$ be a quasi-Prüfer extension, where $T \subseteq S$ is an integral minimal extension and $R \subseteq T$ is integrally closed. Then the diagram (D) is a pullback, $S = T\overline{R}$ and $(T : S) = (R : \overline{R})T$.*

Proof. [5, Lemma 3.5]. \square

Proposition 4.9. *Let $R \subseteq U \subseteq S$ and $R \subseteq V \subseteq S$ be two towers of extensions, such that $R \subseteq U$ and $R \subseteq V$ are almost-Prüfer. Then $R \subseteq UV$ is almost-Prüfer and $\widehat{UV} = \widehat{U}\widehat{V}$.*

Proof. Denote by U', V' and W' the Prüfer hulls of R in U, V and $W = UV$. We deduce from [26, Corollary 5.11, p.53], that $R \subseteq U'V'$ is Prüfer. Moreover, $U'V' \subseteq UV$ is clearly integral and $U'V' \subseteq W'$ because the Prüfer hull is the greatest Prüfer subextension. We deduce that $R \subseteq UV$ is almost-Prüfer and that $\widehat{UV} = \widehat{U}\widehat{V}$. \square

Proposition 4.10. *Let $R \subseteq U \subseteq S$ and $R \subseteq V \subseteq S$ be two towers of extensions, such that $R \subseteq U$ is almost-Prüfer and $R \subseteq V$ is a flat epimorphism. Then $U \subseteq UV$ is almost-Prüfer.*

Proof. Mimic the proof of Proposition 4.9 and use [26, Theorem 5.10, p.53]. \square

Proposition 4.11. *Let $R \subseteq S$ be an almost-Prüfer extension and $R \rightarrow T$ a flat epimorphism. Then $T \subseteq T \otimes_R S$ is almost-Prüfer.*

Proof. It is enough to use Proposition 3.10 and Definition 4.1. \square

Proposition 4.12. *An extension $R \subseteq S$ is almost-Prüfer if and only if $R_P \subseteq S_P$ is almost-Prüfer and $\widetilde{R}_P = (\widetilde{R})_P$ for each $P \in \text{Spec}(R)$.*

Proof. For an arbitrary extension $R \subseteq S$ we have $(\widetilde{R})_P \subseteq \widetilde{R}_P$. Suppose that $R \subseteq S$ is almost-Prüfer, then so is $R_P \subseteq S_P$ and $(\widetilde{R})_P = \widetilde{R}_P$ by Theorem 4.4. Conversely, if $R \subseteq S$ is locally almost-Prüfer, whence locally quasi-Prüfer, then $R \subseteq S$ is quasi-Prüfer. If $\widetilde{R}_P = (\widetilde{R})_P$ holds for each $P \in \text{Spec}(R)$, we have $S_P = (\overline{R}\widetilde{R})_P$ so that $S = \overline{R}\widetilde{R}$ and $R \subseteq S$ is almost-Prüfer by Theorem 4.6. \square

Corollary 4.13. *An FCP extension $R \subseteq S$ is almost-Prüfer if and only if $R_P \subseteq S_P$ is almost-Prüfer for each $P \in \text{Spec}(R)$.*

Proof. It is enough to show that $R \subseteq S$ is almost-Prüfer if $R_P \subseteq S_P$ is almost-Prüfer for each $P \in \text{Spec}(R)$ using Proposition 4.12. Any minimal extension $\widetilde{R} \subset R_1$ is integral by definition of \widetilde{R} . Assume that $(\widetilde{R})_P \subset (\widetilde{R}_P)$, so that there exists $R'_2 \in [\widetilde{R}, S]$ such that $(\widetilde{R})_P \subset (R'_2)_P$ is a Prüfer minimal extension with crucial maximal ideal $Q(\widetilde{R})_P$, for some $Q \in \text{Max}(\widetilde{R})$ with $Q \cap R \subseteq P$. In particular, $\widetilde{R} \subset R'_2$ is not integral. We may assume that there exists $R'_1 \in [\widetilde{R}, R'_2]$ such that $R'_1 \subset R'_2$ is a Prüfer minimal extension with $P \notin \text{Supp}(R'_1/\widetilde{R})$. Using [39, Lemma 1.10], there exists $R_2 \in [\widetilde{R}, R'_2]$ such that $\widetilde{R} \subset R_2$ is a Prüfer minimal extension with crucial maximal ideal Q , a contradiction. Then, $(\widetilde{R})_P \subset S_P$ is integral for each P , whence $(\widetilde{R})_P = (\widetilde{R}_P)$. \square

We now intend to demonstrate that our methods allow us to prove easily some results. For instance, next statement generalizes [5, Corollary 4.5] and can be fruitful in algebraic number theory.

Proposition 4.14. *Let (R, M) be a one-dimensional local ring and $R \subseteq S$ a quasi-Prüfer extension. Suppose that there is a tower $R \subset T \subseteq S$, where $R \subset T$ is integrally closed. Then $R \subseteq S$ is almost-Prüfer, $T = \widetilde{R}$ and S is zero-dimensional.*

Proof. Because $R \subset T$ is quasi-Prüfer and integrally closed, it is Prüfer. If some prime ideal of T is lying over M , $R \subset T$ is a faithfully flat epimorphism, whence an isomorphism by Scholium A, which is absurd. Now let N be a prime ideal of T and $P := N \cap R$. Then R_P is zero-dimensional and isomorphic to T_N . Therefore, T is zero-dimensional. It follows that $T\overline{R}$ is zero-dimensional. Since $R\overline{T} \subseteq S$ is Prüfer, we deduce from Scholium A, that $\overline{R}T = S$. The proof is now complete. \square

We also generalize [5, Proposition 5.2] as follows.

Proposition 4.15. *Let $R \subset S$ be a quasi-Prüfer extension, such that \overline{R} is local with maximal ideal $N := \sqrt{(R : \overline{R})}$. Then R is local and $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$. If in addition R is one-dimensional, then either $R \subset S$ is integral or there is some minimal prime ideal P of \overline{R} , such that $S = (\overline{R})_P$, $P = SP$ and \overline{R}/P is a one-dimensional valuation domain with quotient field S/P .*

Proof. R is obviously local. Let $T \in [R, S] \setminus [R, \overline{R}]$ and $s \in T \setminus \overline{R}$. Then $s \in U(S)$ and $s^{-1} \in \overline{R}$ by Proposition 1.2 (1). But $s^{-1} \notin U(\overline{R})$, so that $s^{-1} \in N$. It follows that there exists some integer n such that $s^{-n} \in (R : \overline{R})$, giving $s^{-n}\overline{R} \subseteq R$, or, equivalently, $\overline{R} \subseteq Rs^n \subseteq T$. Then, $T \in [\overline{R}, S]$ and we obtain $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$.

Assume that R is one-dimensional. If $R \subset S$ is not integral then $\overline{R} \subset S$ is Prüfer and \overline{R} is one-dimensional. To complete the proof, use Proposition 1.2 (3). \square

4.2. FCP extensions. In case we consider only FCP extensions, we obtain more results.

Proposition 4.16. *Let $R \subseteq S$ be an FCP extension. The following statements are equivalent:*

- (1) $R \subseteq S$ is almost-Prüfer.
- (2) $R_P \subseteq S_P$ is either integral or Prüfer for each $P \in \text{Spec}(R)$.
- (3) $R_P \subseteq S_P$ is almost-Prüfer and $\text{Supp}(S/\tilde{R}) \cap \text{Supp}(\tilde{R}/R) = \emptyset$.
- (4) $\text{Supp}(\overline{R}/R) \cap \text{Supp}(S/\overline{R}) = \emptyset$.

Proof. The equivalence of Proposition 4.12 shows that (2) \Leftrightarrow (1) holds because $\widehat{T} = \widetilde{T}$ and over a local ring T , an almost-Prüfer FCP extension $T \subseteq U$ is either integral or Prüfer [39, Proposition 2.4]. Moreover when $R_P \subseteq S_P$ is either integral or Prüfer, it is easy to show that $(\tilde{R})_P = \widetilde{R}_P$.

Next we show that (3) is equivalent to (2) of Proposition 4.12.

Let $P \in \text{Supp}(S/\tilde{R}) \cap \text{Supp}(\tilde{R}/R)$ be such that $R_P \subseteq S_P$ is almost-Prüfer. Then, $(\tilde{R})_P \neq R_P, S_P$, so that $R_P \subset (\tilde{R})_P \subset S_P$. Since $R \subset \tilde{R}$ is Prüfer, so is $R_P \subset (\tilde{R})_P$, giving $(\tilde{R})_P \subseteq \widetilde{R}_P$ and $R_P \neq \widetilde{R}_P$. It follows that $\widetilde{R}_P = S_P$ in view of the dichotomy principle [39, Proposition 3.3] since R_P is a local ring, and then $\widetilde{R}_P \neq (\tilde{R})_P$.

Conversely, assume that $\widetilde{R}_P \neq (\tilde{R})_P$, i.e. $P \in \text{Supp}(S/R)$. Then, $R_P \neq \widetilde{R}_P$, so that $\widetilde{R}_P = S_P$, as we have just seen. Hence $R_P \subset S_P$ is integrally closed. It follows that $\overline{R}_P = \overline{R}_P = R_P$, so that $P \notin \text{Supp}(\overline{R}/R)$ and $P \in \text{Supp}(\tilde{R}/R)$ by Theorem 4.6(5). Moreover, $\widetilde{R}_P \neq$

S_P implies that $P \in \text{Supp}(S/\tilde{R})$. To conclude, $P \in \text{Supp}(S/\tilde{R}) \cap \text{Supp}(\tilde{R}/R)$.

(1) \Leftrightarrow (4) An FCP extension is quasi-Prüfer by Corollary 3.4. Suppose that $R \subseteq S$ is almost-Prüfer. By Theorem 4.6, letting $U := \tilde{R}$, we get that $U \cap \overline{R} = R$ and $S = \overline{R}U$. We deduce from [39, Proposition 3.6] that $\text{Supp}(\overline{R}/R) \cap \text{Supp}(S/\overline{R}) = \emptyset$. Suppose that this last condition holds. Then by [39, Proposition 3.6] $R \subseteq S$ can be factored $R \subseteq U \subseteq S$, where $R \subseteq U$ is integrally closed, whence Prüfer by Proposition 1.3, and $U \subseteq S$ is integral. Therefore, $R \subseteq S$ is almost-Prüfer. \square

Lemma 4.17. *Let $B \subset D$ and $C \subset D$ be two integral minimal extensions and $A := B \cap C$. If $A \subset D$ has FCP, then, $A \subset D$ is integral.*

Proof. Set $M := (B : D)$ and $N := (C : D)$.

If $M \neq N$, then, $A \subset D$ is integral by [13, Proposition 6.6].

Assume that $M = N$. Then, $M \in \text{Max}(A)$ by [13, Proposition 5.7]. Let B' be the integral closure of A in B . Then M is also an ideal of B' , which is prime in B' , and then maximal in B' . If $A \subset D$ is an FCP extension, so is $B' \subseteq B$, which is a flat epimorphism, and so is $B'/M \subseteq B/M$. Then, $B' = B$ since B'/M is a field. It follows that $A \subseteq B$ is an integral extension, and so is $A \subset D$. \square

Proposition 4.18. *Let $R \subset S$ be an FCP almost-Prüfer extension. Then, $\tilde{R} = \hat{R}$ is the least $T \in [R, S]$ such that $T \subseteq S$ is integral.*

Proof. We may assume that $R \subset S$ is not integral. If there is some $U \in [R, \tilde{R}]$ such that $U \subseteq \tilde{R}$ is integral, then $U = \tilde{R}$. Set $X := \{T \in [R, S] \mid T \subseteq S \text{ integral}\}$. It follows that \tilde{R} is a minimal element of X . We are going to show that \tilde{R} is the least element of X .

Set $n := \ell[\tilde{R}, S] \geq 1$ and let $\tilde{R} = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$ be a maximal chain of $[\tilde{R}, S]$, with length n . There does not exist a maximal chain of \tilde{R} -subalgebras of S with length $> n$. Let $T \in X$. We intend to show that $T \in [\tilde{R}, S]$. It is enough to choose T such that T is a minimal element of X . Consider the induction hypothesis: (H_n) : $X \subseteq [\tilde{R}, S]$ when $n := \ell[\tilde{R}, S]$.

We first show (H_1) . If $n = 1$, $\tilde{R} \subset S$ is minimal. Let $T \in X$ and $T_1 \in [T, S]$ be such that $T_1 \subset S$ is minimal. Assume that $T_1 \neq \tilde{R}$. Lemma 4.17 shows that $T_1 \cap \tilde{R} \subset \tilde{R}$ is integral, which contradicts the beginning of the proof. Then, $T_1 = \tilde{R}$, so that $T = \tilde{R}$ for the same contradiction and (H_1) is proved.

Assume that $n > 1$ and that (H_k) holds for any $k < n$. Let $T \in X$ and $T_1 \in [T, S]$ be such that $T_1 \subset S$ is minimal. If $T_1 \in [\tilde{R}, S]$, then

$k := \ell[\tilde{R}, T_1] \leq n-1$. But we get that $T \in [R, T_1]$, with $T \subseteq T_1$ integral. Moreover, \tilde{R} is also the Prüfer hull of $R \subseteq T_1$, with $k := \ell[\tilde{R}, T_1] \leq n-1$. Since (H_k) holds, we get that $T \in [\tilde{R}, T_1] \subset [\tilde{R}, S]$.

If $T_1 \notin [\tilde{R}, S]$, set $U := T_1 \cap R_{n-1}$. We get that $T_1 \subset S$ and $R_{n-1} \subset S$ are minimal and integral. Using again Lemma 4.17, we get that $U \subseteq S$ is integral, with $\ell[\tilde{R}, R_{n-1}] = n-1$ and $U \in [R, R_{n-1}]$. As before, \tilde{R} is also the Prüfer hull of $R \subseteq R_{n-1}$. Since (H_{n-1}) holds, $U \in [\tilde{R}, R_{n-1}]$, so that $T_1 \in [\tilde{R}, S]$, a contradiction. Therefore, (H_n) is proved. \square

We will need a relative version of the support. Let $f : R \rightarrow T$ be a ring morphism and E a T -module. The relative support of E over R is $\mathcal{S}_R(E) := {}^a f(\text{Supp}_T(E))$ and $\text{MS}_R(E) := \mathcal{S}_R(E) \cap \text{Max}(R)$. In particular, for a ring extension $R \subset S$, we have $\mathcal{S}_R(S/R) := \text{Supp}_R(S/R)$.

Proposition 4.19. *Let $R \subseteq S$ be an FCP extension. The following statements hold:*

- (1) $\text{Supp}(\tilde{\tilde{R}}/\tilde{R}) \cap \text{Supp}(\tilde{R}/R) = \emptyset$.
- (2) $\text{Supp}(\tilde{R}/R) \cap \text{Supp}(\tilde{R}/R) = \text{Supp}(\tilde{\tilde{R}}/\tilde{R}) \cap \text{Supp}(\tilde{R}/R) = \emptyset$.
- (3) $\text{MSupp}(S/R) = \text{MSupp}(\tilde{R}/R) \cup \text{MSupp}(\tilde{R}/R)$.

Proof. (1) is a consequence of Proposition 4.16(4) because $R \subseteq \tilde{\tilde{R}}$ is almost-Prüfer.

We prove the first part of (2). If some $M \in \text{Supp}(\tilde{R}/R) \cap \text{Supp}(\tilde{R}/R)$, it can be supposed in $\text{Max}(R)$. Set $R' := R_M, U := (\tilde{R})_M, T := (\tilde{R})_M$ and $M' := MR_M$. Then, $R' \neq U, T$, with $R' \subset U$ FCP Prüfer and $R' \subset T$ FCP integral, an absurdity [39, Proposition 3.3].

To show the second part, assume that some $P \in \text{Supp}(\tilde{\tilde{R}}/\tilde{R}) \cap \text{Supp}(\tilde{R}/R)$. Then, $P \notin \text{Supp}(\tilde{R}/R)$ by the first part of (2), so that $\tilde{R}_P = R_P$, giving $(\tilde{\tilde{R}})_P = \tilde{R}_P \tilde{R}_P = \tilde{R}_P$, a contradiction.

(3) Obviously, $\text{MSupp}(S/R) = \text{MS}_R(S/R) = \text{MS}_R(S/\tilde{T}^S) \cup \text{MS}_R(\tilde{T}^S/T) \cup \text{MS}_R(T/\tilde{U}^T) \cup \text{MS}_R(\tilde{U}^T/U) \cup \text{MS}_R(U/R)$. By [39, Propositions 2.3 and 3.2], we have $\text{MS}_R(S/\tilde{T}^S) \subseteq \mathcal{S}(\tilde{T}^S/T) = \mathcal{S}(\tilde{R}/\tilde{R}^T) = \text{MS}_R(\tilde{R}/R) = \text{MSupp}(\tilde{R}/R)$, $\text{MS}_R(T/\tilde{U}^T) = \mathcal{S}(\tilde{R}^T/R) \subseteq \mathcal{S}(\tilde{R}/R) = \text{Supp}(\tilde{R}/R)$ and $\text{MS}_R(\tilde{U}^T/U) = \mathcal{S}(\tilde{R}^T/R) = \text{Supp}(\tilde{R}/R)$. To conclude, $\text{MSupp}(S/R) = \text{MSupp}(\tilde{R}/R) \cup \text{MSupp}(\tilde{R}/R)$. \square

Proposition 4.20. *Let $R \subset S$ be an FCP extension and $M \in \text{MSupp}(S/R)$, then $\tilde{\tilde{R}}_M = (\tilde{R})_M$ if and only if $M \notin \text{MSupp}(S/\tilde{R}) \cap \text{MSupp}(\tilde{R}/R)$.*

Proof. In fact, we are going to show that $\tilde{\tilde{R}}_M \neq (\tilde{R})_M$ if and only if $M \in \text{MSupp}(S/\tilde{R}) \cap \text{MSupp}(\tilde{R}/R)$.

Let $M \in \text{MSupp}(S/\tilde{R}) \cap \text{MSupp}(\tilde{R}/R)$. Then, $\widetilde{R_M} \neq R_M, S_M$ and then $R_M \subset \widetilde{R_M} \subset S_M$. Since $R \subset \tilde{R}$ is Prüfer, so is $R_M \subset \widetilde{R_M}$ by Proposition 1.2, giving $(\tilde{R})_M \subseteq \widetilde{R_M}$ and $R_M \neq \widetilde{R_M}$. Therefore, $\widetilde{R_M} = S_M$ [39, Proposition 3.3] since R_M is local, and then $\widetilde{R_M} \neq (\tilde{R})_M$.

Conversely, if $\widetilde{R_M} \neq (\tilde{R})_M$, then, $R_M \neq \widetilde{R_M}$, so that $\widetilde{R_M} = S_M$, as we have just seen and then $R_M \subset S_M$ is integrally closed. It follows that $\widetilde{R_M} = \overline{R_M} = R_M$, so that $M \notin \text{MSupp}(\tilde{R}/R)$. Hence, $M \in \text{MSupp}(\tilde{R}/R)$ by Proposition 4.19(3). Moreover, $\tilde{R_M} \neq S_M \Rightarrow M \in \text{MSupp}(S/\tilde{R})$. To conclude, $M \in \text{MSupp}(S/\tilde{R}) \cap \text{MSupp}(\tilde{R}/R)$. \square

If $R \subseteq S$ is any ring extension, with $\dim(R) = 0$, then $\widetilde{R_M} = (\tilde{R})_M$ for any $M \in \text{Max}(R)$. Indeed by Scholium A (2), the flat epimorphism $R \rightarrow \tilde{R}$ is bijective as well as $R_M \rightarrow (\tilde{R})_M$. This conclusion is still valid in another context.

Corollary 4.21. *Let $R \subset S$ be an FCP extension. Assume that one of the following conditions is satisfied:*

- (1) $\text{MSupp}(S/\tilde{R}) \cap \text{MSupp}(\tilde{R}/R) = \emptyset$.
- (2) $S = \overline{R\tilde{R}}$, or equivalently, $R \subseteq S$ is almost-Prüfer.

Then, $\widetilde{R_M} = (\tilde{R})_M$ for any $M \in \text{Max}(R)$.

Proof. (1) is Proposition 4.20. (2) is Proposition 4.12. \square

Proposition 4.22. *Let $R \subset S$ be an almost-Prüfer FCP extension. Then, any $T \in [R, S]$ is the integral closure of $T \cap \tilde{R}$ in $T\tilde{R}$.*

Proof. Set $U := T \cap \tilde{R}$ and $V := T\tilde{R}$. Since $R \subset S$ is almost-Prüfer, $U \subseteq \tilde{R}$ is Prüfer and $\tilde{R} \subseteq V$ is integral and \tilde{R} is also the Prüfer hull of $U \subseteq V$. Because $R \subset S$ is almost-Prüfer, for each $M \in \text{MSupp}_R(S/R)$, $R_M \subseteq S_M$ is either integral, or Prüfer by Proposition 4.16, and so is $U_M \subseteq V_M$. But $\widetilde{R_M} = (\tilde{R})_M$ by Corollary 4.21 is also the Prüfer hull of $U_M \subseteq V_M$. Let T' be the integral closure of U in V . Then, T'_M is the integral closure of U_M in V_M .

Assume that $U_M \subseteq V_M$ is integral. Then $V_M = T'_M$ and $U_M = (\tilde{R})_M$, so that $V_M = T_M(\tilde{R})_M = T_M$, giving $T_M = T'_M$.

Assume that $U_M \subseteq V_M$ is Prüfer. Then $U_M = T'_M$ and $V_M = (\tilde{R})_M$, so that $U_M = T_M \cap (\tilde{R})_M = T_M$, giving $T_M = T'_M$.

To conclude, we get that $T_M = T'_M$ for each $M \in \text{MSupp}_R(S/R)$. Since $R_M = S_M$, with $T_M = T'_M$ for each $M \in \text{Max}(R) \setminus \text{MSupp}_R(S/R)$, we get $T = T'$, whence T is the integral closure of $U \subseteq V$. \square

We build an example of an FCP extension $R \subset S$ where we have $\widetilde{R}_M \neq (\widetilde{R})_M$ for some $M \in \text{Max}(R)$. In particular, $R \subset S$ is not almost-Prüfer.

Example 4.23. Let R be an integral domain with quotient field S and $\text{Spec}(R) := \{M_1, M_2, P, 0\}$, where $M_1 \neq M_2$ are two maximal ideals and P a prime ideal satisfying $P \subset M_1 \cap M_2$. Assume that there are R_1, R_2 and R_3 such that $R \subset R_1$ is Prüfer minimal, with $\mathcal{C}(R, R_1) = M_1$, $R \subset R_2$ is integral minimal, with $\mathcal{C}(R, R_2) = M_2$ and $R_2 \subset R_3$ is Prüfer minimal, with $\mathcal{C}(R_2, R_3) = M_3 \in \text{Max}(R_2)$ such that $M_3 \cap R = M_2$ and $M_2 R_3 = R_3$. This last condition is satisfied when $R \subset R_2$ is either ramified or inert. Indeed, in both cases, $M_3 R_3 = R_3$; moreover, in the ramified case, we have $M_3^2 \subseteq M_2$ and in the inert case, $M_3 = M_2$ [36, Theorem 3.3]. We apply [13, Proposition 7.10] and [10, Lemma 2.4] several times. Set $R'_2 := R_1 R_2$. Then, $R_1 \subset R'_2$ is integral minimal, with $\mathcal{C}(R_1, R'_2) =: M'_2 = M_2 R_1$ and $R_2 \subset R'_2$ is Prüfer minimal, with $\mathcal{C}(R_2, R'_2) =: M'_1 = M_1 R_2 \in \text{Max}(R_2)$. Moreover, $M'_1 \neq M_3$, $\text{Spec}(R_1) = \{M'_2, P_1, 0\}$, where P_1 is the only prime ideal of R_1 lying over P . But, $P = (R : R_1)$ by [17, Proposition 3.3], so that $P = P_1$. Set $R'_3 := R_3 R'_2$. Then, $R'_2 \subset R'_3$ is Prüfer minimal, with $\mathcal{C}(R'_2, R'_3) =: M'_3 = M_3 R'_2 \in \text{Max}(R'_2)$ and $R_3 \subset R'_3$ is Prüfer minimal, with $\mathcal{C}(R_3, R'_3) = M''_1 = M_1 R_3 \in \text{Max}(R_3)$. It follows that we have $\text{Spec}(R'_3) = \{P', 0\}$ where P' is the only prime ideal of R'_3 lying over P . To end, assume that $R'_3 \subset S$ is Prüfer minimal, with $\mathcal{C}(R'_3, S) = P'$. Hence, R_2 is the integral closure of R in S . In particular, $R \subset S$ has FCP [10, Theorems 6.3 and 3.13] and is quasi-Prüfer. Since $R \subset R_1$ is integrally closed, we have $R_1 \subseteq \widetilde{R}$. Assume that $R_1 \neq \widetilde{R}$. Then, there exists $T \in [R_1, S]$ such that $R_1 \subset T$ is Prüfer minimal and $\mathcal{C}(R_1, T) = M'_2$, a contradiction by Proposition 4.16 since $M'_2 = \mathcal{C}(R_1, R'_2)$, with $R_1 \subset R'_2$ integral minimal. Then, $R_1 = \widetilde{R}$. It follows that $M_1 \in \text{MSupp}(\widetilde{R}/R)$. But, $P = \mathcal{C}(R'_3, S) \cap R \in \text{Supp}(S/\widetilde{R})$ and $P \subset M_1$ give $M_1 \in \text{MSupp}(S/\widetilde{R})$, so that $\widetilde{R}_{M_1} \neq (\widetilde{R})_{M_1}$ by Proposition 4.20 giving that $R \subset S$ is not almost-Prüfer.

We now intend to refine Theorem 4.6, following the scheme used in [4, Proposition 4] for extensions of integral domains.

Proposition 4.24. *Let $R \subseteq S$ and $U, T \in [R, S]$ be such that $R \subseteq U$ is integral and $R \subseteq T$ is Prüfer. Then $U \subseteq UT$ is Prüfer in the following cases and $R \subseteq UT$ is almost-Prüfer.*

- (1) $\text{Supp}(\widetilde{R}/R) \cap \text{Supp}(\widetilde{R}/R) = \emptyset$ (for example, if $R \subseteq S$ has FCP).
- (2) $R \subseteq U$ preserves integral closure.

Proof. (1) We have $\emptyset = \text{MSupp}(U/R) \cap \text{MSupp}(T/R)$, since $U \subseteq \overline{R}$ and $T \subseteq \tilde{R}$. Let $M \in \text{MSupp}((UT)/R)$. For $M \in \text{MSupp}(U/R)$, we have $R_M = T_M$ and $(UT)_M = U_M$. If $M \notin \text{MSupp}(U/R)$, then $U_M = R_M$ and $(UT)_M = T_M$, so that $U_M \subseteq (UT)_M$ identifies to $R_M \subseteq T_M$.

Let $N \in \text{Max}(U)$ and set $M := N \cap R \in \text{Max}(R)$ since $R \subseteq U$ is integral. If $M \notin \text{Supp}(\overline{R}/R)$, then $R_M = \overline{R}_M = U_M$ and N is the only maximal ideal of U lying over M . It follows that $U_M = U_N$ and $(UT)_M = (UT)_N$ by [10, Lemma 2.4]. Then, $U_N \subseteq (UT)_N$ identifies to $R_M \subseteq T_M$ which is Prüfer. If $M \notin \text{Supp}(\tilde{R}/R)$, then $R_M = T_M$ gives $U_M = (UT)_M$, so that $U_N = (UT)_N$ by localizing the precedent equality and $U_N \subseteq (UT)_N$ is still Prüfer. Therefore, $U \subseteq UT$ is locally Prüfer, whence Prüfer by Proposition 1.1.

(2) The usual reasoning shows that $U \otimes_R T \cong UT$, so that $U \subseteq UT$ is integrally closed. Since U is contained in \overline{R}^{UT} , we get that $U = \overline{R}^{UT}$. Now observe that $R \subseteq UT$ is almost-Prüfer, whence quasi-Prüfer. It follows that $U \subseteq UT$ is Prüfer. \square

Next propositions generalize Ayache's results of [4, Proposition 11].

Proposition 4.25. *Let $R \subseteq S$ be a quasi-Prüfer extension, $T, T' \in [R, S]$ and $U := T \cap T'$. The following statements hold:*

- (1) $\tilde{T} = \widetilde{(T \cap \overline{R})}$ for each $T \in [R, S]$.
- (2) $\tilde{T} \cap \tilde{T}' \subseteq \widetilde{T \cap T'}$.
- (3) Let $\text{Supp}(\overline{T}/T) \cap \text{Supp}(\tilde{T}/T) = \emptyset$ (this assumption holds if $R \subseteq S$ has FCP). Then, $T \subseteq T' \Rightarrow \tilde{T} \subseteq \tilde{T}'$.
- (4) If $\text{Supp}(\overline{U}/U) \cap \text{Supp}(\tilde{U}/U) = \emptyset$, then $\tilde{T} \cap \tilde{T}' = \widetilde{T \cap T'}$.

Proof. (1) We observe that $R \subseteq T$ is quasi-Prüfer by Corollary 3.3. Since $T \cap \overline{R}$ is the integral closure of R in T , we get that $T \cap \overline{R} \subseteq T$ is Prüfer. It follows that $T \cap \overline{R} \subseteq \tilde{T}$ is Prüfer. We thus have $\tilde{T} \subseteq \widetilde{T \cap \overline{R}}$. To prove the reverse inclusion, we set $V := T \cap \overline{R}$ and $W := \tilde{V} \cap \tilde{T}$. We have $W \cap \overline{R} = \tilde{V} \cap \overline{R} = V$, because $V \subseteq \tilde{V} \cap \overline{R}$ is integral and Prüfer since we have a tower $V \subseteq \tilde{V} \cap \overline{R} \subseteq \tilde{V}$. Therefore, $V \subseteq W$ is Prüfer because $W \in [V, \tilde{V}]$. Moreover, $T \subseteq \tilde{T} \subseteq \tilde{V}$, since $V \subseteq \tilde{T}$ is Prüfer. Then, $T \subseteq W$ is integral because $W \in [T, \tilde{T}]$, and we have $V \subseteq T \subseteq W$. This entails that $T = W = \tilde{V} \cap \overline{R}$, so that $T \subseteq \tilde{V}$ is Prüfer. It follows that $\tilde{V} \subseteq \tilde{T}$ since $T \in [V, \tilde{V}]$.

(2) A quasi-Prüfer extension is Prüfer if and only if it is integrally closed. We observe that $T \cap T' \subseteq \tilde{T} \cap \tilde{T}'$ is integrally closed, whence Prüfer. It follows that $\tilde{T} \cap \tilde{T}' \subseteq \widetilde{T \cap T'}$.

(3) Set $U = T \cap \overline{R}$ and $U' = T' \cap \overline{R}$, so that $U, U' \in [R, \overline{R}]$ with $U \subseteq U'$. In view of (1), we thus can suppose that $T, T' \in [R, \overline{R}]$. It follows that $T \subseteq T'$ is integral and $T \subseteq \widetilde{T}$ is Prüfer. We deduce from Proposition 4.24(1) that $T' \subseteq T'\widetilde{T}$ is Prüfer, so that $\widetilde{T}T' \subseteq \widetilde{T}'$, because $\text{Supp}(\overline{T}/T) \cap \text{Supp}(\widetilde{T}/T) = \emptyset$ and $\overline{T} = \overline{R}$. Therefore, we have $\widetilde{T} \subseteq \widetilde{T}'$.

(4) Assume that $\text{Supp}(\overline{U}/U) \cap \text{Supp}(\widetilde{U}/U) = \emptyset$. Then, $T \cap T' \subset T, T'$ gives $\widetilde{T \cap T'} \subseteq \widetilde{T} \cap \widetilde{T}'$ in view of (3), so that $\widetilde{T \cap T'} = \widetilde{T} \cap \widetilde{T}'$ by (2). \square

Proposition 4.26. *Let $R \subseteq S$ be a quasi-Prüfer extension and $T \subseteq T'$ a subextension of $R \subseteq S$. Set $U := T \cap \overline{R}$, $U' := T' \cap \overline{R}$, $V := T\overline{R}$ and $V' := T'\overline{R}$. The following statements hold:*

- (1) $T \subseteq T'$ is integral if and only if $V = V'$.
- (2) $T \subseteq T'$ is Prüfer if and only if $U = U'$.
- (3) Assume that $U \subset U'$ is integral minimal and $V = V'$. Then, $T \subset T'$ is integral minimal, of the same type as $U \subset U'$.
- (4) Assume that $V \subset V'$ is Prüfer minimal and $U = U'$. Then, $T \subset T'$ is Prüfer minimal.
- (5) Assume that $T \subset T'$ is minimal and set $P := \mathcal{C}(T, T')$.
 - (a) If $T \subset T'$ is integral, then $U \subset U'$ is integral minimal if and only if $P \cap U \in \text{Max}(U)$.
 - (b) If $T \subset T'$ is Prüfer, then $V \subset V'$ is Prüfer minimal if and only if there is exactly one prime ideal in V lying over P .

Proof. In $[R, S]$ we have the integral extensions $U \subseteq U'$, $T \subseteq V$, $T' \subseteq V'$ and the Prüfer extensions $V \subseteq V'$, $U \subseteq T$, $U' \subseteq T'$. Moreover, \overline{R} is also the integral closure of $U \subseteq V'$.

(1) is gotten by considering the extension $T \subseteq V'$, which is both $T \subseteq V \subseteq V'$ and $T \subseteq T' \subseteq V'$.

(2) is gotten by considering the extension $U \subseteq T'$, which is both $U \subseteq T \subseteq T'$ and $U \subseteq U' \subseteq T'$.

(3) Assume that $U \subset U'$ is integral minimal and $V = V'$. Then, $T \subset T'$ is integral by (1) and $T \neq T'$ because of (2). Set $M := (U : U') \in \text{Supp}_U(U'/U)$. For any $M' \in \text{Max}(U)$ such that $M' \neq M$, we have $U_{M'} = U'_{M'}$, so that $T_{M'} = T'_{M'}$ because $U_{M'} \subseteq T'_{M'}$ is Prüfer. But, $U \subseteq T'$ is almost-Prüfer, giving $T' = TU'$. By Theorem 4.6, $(T : T') = (U : U')T = MT \neq T$ because $T \neq T'$. We get that $U \subseteq T$ Prüfer implies that $M \notin \text{Supp}_U(T/U)$ and $U_M = T_M$. It follows that $T'_M = T_M U'_M = U'_M$. Therefore, $T_M \subseteq T'_M$ identifies to $U_M \subseteq U'_M$, which is minimal of the same type as $U \subset U'$ by [13, Proposition 4.6]. Then, $T \subset T'$ is integral minimal, of the same type as $U \subset U'$.

(4) Assume that $V \subset V'$ is Prüfer minimal and $U = U'$. Then, $T \subset T'$ is Prüfer by (2) and $T \neq T'$ because of (1). Set $Q := \mathcal{C}(V, V')$

and $P := Q \cap T \in \text{Max}(T)$ since $Q \in \text{Max}(V)$. For any $P' \in \text{Max}(T)$ such that $P' \neq P$, and $Q' \in \text{Max}(V)$ lying above P' , we have $V_{Q'} = V'_{Q'}$, so that $V_{P'} = V'_{P'}$. It follows that $T'_{P'} \subseteq V'_{P'}$ is integral, so that $T_{P'} = T'_{P'}$ and $P' \notin \text{Supp}_T(T'/T)$. We get that $T \subset T'$ is Prüfer minimal in view of [10, Proposition 6.12].

(5) Assume that $T \subset T'$ is a minimal extension and set $P := \mathcal{C}(T, T')$.

(a) Assume that $T \subset T'$ is integral. Then, $V = V'$ and $U \neq U'$ by (1) and (2). We can use Proposition 4.5 getting that $P = (U : U')T \in \text{Max}(T)$ and $Q := (U : U') = P \cap U \in \text{Spec}(U)$. It follows that $Q \notin \text{Supp}_U(T/U)$, so that $U_Q = T_Q$ and $U'_Q = T'_Q$. Then, $U_Q \subset U'_Q$ is integral minimal, with $Q \in \text{Supp}_U(U'/U)$.

If $Q \notin \text{Max}(U)$, then $U \subset U'$ is not minimal by the properties of the crucial maximal ideal.

Assume that $Q \in \text{Max}(U)$ and let $M \in \text{Max}(U)$, with $M \neq Q$. Then, $U_M = U'_M$ because $M + Q = U$, so that $U \subset U'$ is a minimal extension and (a) is gotten.

(b) Assume that $T \subset T'$ is Prüfer. Then, $V \neq V'$ and $U = U'$ by (1) and (2). Moreover, $PT' = T'$ gives $PV' = V'$. Let $Q \in \text{Max}(V)$ lying over P . Then, $QV' = V'$ gives that $Q \in \text{Supp}_V(V'/V)$. Moreover, we have $V' = VT'$. Let $P' \in \text{Max}(T)$, $P' \neq P$. Then, $T_{P'} = T'_{P'}$ gives $V_{P'} = V'_{P'}$. It follows that $\text{Supp}_T(V'/V) = \{P\}$ and $\text{Supp}_V(V'/V) = \{Q \in \text{Max}(V) \mid Q \cap T = P\}$. But, by [10, Proposition 6.12], $V \subset V'$ is Prüfer minimal if and only if $|\text{Supp}_V(V'/V)| = 1$, and then if and only if there is exactly one prime ideal in V lying over P . \square

Lemma 4.27. *Let $R \subseteq S$ be an FCP almost-Prüfer extension and $U \in [R, \overline{R}]$, $V \in [\overline{R}, S]$. Then $U \subseteq V$ has FCP and is almost-Prüfer.*

Proof. Obviously, $U \subseteq V$ has FCP and \overline{R} is the integral closure of U in V . Proposition 4.16 entails that $\text{Supp}_R(\overline{R}/R) \cap \text{Supp}_R(S/\overline{R}) = \emptyset$. We claim that $\text{Supp}_U(\overline{R}/U) \cap \text{Supp}_U(V/\overline{R}) = \emptyset$. Deny and let $Q \in \text{Supp}_U(\overline{R}/U) \cap \text{Supp}_U(V/\overline{R})$. Then, $\overline{R}_Q \neq U_Q, V_Q$. If $P := Q \cap R$, we get that $\overline{R}_P \neq U_P, V_P$, giving $\overline{R}_P \neq R_P, S_P$, a contradiction. Another use of Proposition 4.16 shows that $U \subseteq V$ is almost-Prüfer. \square

Proposition 4.28. *Let $R \subseteq S$ be an FCP almost-Prüfer extension and $T \subseteq T'$ a subextension of $R \subseteq S$. Set $U := T \cap \overline{R}$ and $V' := T'\overline{R}$. Let W be the Prüfer hull of $U \subseteq V'$. Then, W is also the Prüfer hull of $T \subseteq T'$ and $T \subseteq T'$ is an FCP almost-Prüfer extension.*

Proof. By Lemma 4.27, we get that $U \subseteq V'$ is an FCP almost-Prüfer extension. Let \tilde{T} be the Prüfer hull of $T \subseteq T'$. Since $U \subseteq T$ and $T \subseteq \tilde{T}$ are Prüfer, so is $U \subseteq \tilde{T}$ and $\tilde{T} \subseteq V'$ gives that $\tilde{T} \subseteq W$. Then, $T \subseteq W$ is Prüfer as a subextension of $U \subseteq W$.

Moreover, in view of Proposition 4.18, W is the least U -subalgebra of V' over which V' is integral. Since $T' \subseteq V'$ is integral, we get that $W \subseteq T'$, so that $W \in [T, T']$, with $W \subseteq T'$ integral as a subextension of $W \subseteq V'$. It follows that W is also the Prüfer hull of $T \subseteq T'$ and $T \subseteq T'$ is an FCP almost-Prüfer extension. \square

5. THE CASE OF NAGATA EXTENSIONS

In this section we transfer the quasi-Prüfer (and almost-Prüfer) properties to Nagata extensions.

Proposition 5.1. *Let $R \subseteq S$ be a Prüfer (and FCP) extension, then $R(X) \subseteq S(X)$ is a Prüfer (and FCP) extension.*

Proof. We can suppose that (R, M) is local, in order to use Proposition 1.2(3). Then it is enough to know the following facts: $V(X)$ is a valuation domain if so is V ; $R[X]_{P[X]} \cong R(X)_{P(X)} \cong R_P(X)$ where $P(X) = PR(X)$ and $R(X)/P(X) \cong (R/P)(X)$ for $P \in \text{Spec}(R)$. If in addition $R \subseteq S$ is FCP, it is enough to use [11, Theorem 3.9]: $R \subset S$ has FCP if and only if $R(X) \subset S(X)$ has FCP. \square

Proposition 5.2. *If $R \subseteq S$ is quasi-Prüfer, then so is $R(X) \subseteq S(X)$, $\overline{R(X)} = \overline{R}(X) \cong \overline{R} \otimes_R R(X)$ and $S(X) \cong S \otimes_R R(X)$.*

Proof. It is enough to use proposition 5.1, because $\overline{R(X)} = \overline{R}(X)$. The third assertion results from [34, Proposition 4 and Proposition 7]. \square

Proposition 5.3. *If $R \subseteq S$ is almost-Prüfer, then so is $R(X) \subseteq S(X)$. It follows that $\widetilde{R(X)} = \widetilde{R}(X)$ for an almost-Prüfer extension $R \subseteq S$.*

Proof. If $R \subseteq S$ is almost-Prüfer, then $R \subseteq \widetilde{R}$ is Prüfer and $\widetilde{R} \subseteq S$ is integral and then $R(X) \subseteq \widetilde{R}(X)$ is Prüfer and $\widetilde{R}(X) \subseteq S(X)$ is integral, whence $R(X) \subseteq S(X)$ is almost-Prüfer with $\widetilde{R(X)} = \widetilde{R}(X)$. \square

Lemma 5.4. *Let $R \subset S$ be an FCP ring extension such that $\widetilde{R} = R$. Then, $\widetilde{R(X)} = R(X)$.*

Proof. If $R(X) \neq \widetilde{R(X)}$, there is some $T' \in [R(X), \widetilde{R(X)}]$ such that $R(X) \subset T'$ is Prüfer minimal. Set $\mathcal{C}(R(X), T') \in \text{MSupp}(S(X)/R(X)) =: M'$. There is $M \in \text{MSupp}(S/R)$ such that $M' = MR(X)$ [11, Lemma 3.3]. But, $M' \notin \text{MSupp}(\widetilde{R(X)}/R(X)) = \text{MSupp}(\overline{R}(X)/R(X))$ by Proposition 4.19(2), giving that $M \notin \text{MSupp}(\overline{R}/R) = \mathcal{S}(\overline{R}/R)$. Then [39, Proposition 1.7(3)] entails that $M \in \mathcal{S}(S/\overline{R})$. By [39, Proposition 1.7(4)], there are some $T_1, T_2 \in [\overline{R}, S]$ with $T_1 \subset T_2$ Prüfer minimal (an FCP extension is quasi-Prüfer), with $M = \mathcal{C}(T_1, T_2) \cap R$.

We can choose for $T_1 \subset T_2$ the first minimal extension verifying the preceding property. Therefore, $M \notin \mathcal{S}(T_1/\overline{R})$, so that $M \notin \mathcal{S}(T_1/R) = \text{Supp}(T_1/R)$. By [39, Lemma 1.10], we get that there exists $T \in [R, T_2]$ such that $R \subset T$ is Prüfer minimal, a contradiction. \square

Proposition 5.5. *If $R \subset S$ is an FCP extension, then, $\widetilde{R}(X) = \widetilde{\widetilde{R}(X)}$.*

Proof. Because $R \subseteq \widetilde{R}$ is Prüfer, $R(X) \subseteq \widetilde{R}(X)$ is Prüfer by Corollary 5.1. Then, $\widetilde{R}(X) \subseteq \widetilde{\widetilde{R}(X)}$. Assume that $\widetilde{R}(X) \neq \widetilde{\widetilde{R}(X)}$ and set $T := \widetilde{R}$, so that $T = \widetilde{T}$, giving $\widetilde{T(X)} = T(X) = \widetilde{R}(X)$ by Lemma 5.4. Hence $\widetilde{T(X)} \subset \widetilde{\widetilde{R}(X)}$ is a Prüfer extension, contradicting the definition of $\widetilde{\widetilde{R}(X)}$. So, $\widetilde{R}(X) = \widetilde{\widetilde{R}(X)}$. \square

Proposition 5.6. *Let $R \subseteq S$ be an almost-Prüfer FCP extension, then $\widehat{R}(X) = \widehat{\widetilde{R}(X)} = \widetilde{R(X)}$.*

Proof. We have a tower $R(X) \subseteq \widehat{R(X)} = \widehat{\widetilde{R(X)}} = \widetilde{R(X)} = \widehat{R(X)}$, where the first and the third equalities come from Theorem 4.4 and the second from Proposition 5.5. \square

We end this section with a special result.

Proposition 5.7. *Let $R \subseteq S$ be an extension such that $R(X) \subseteq S(X)$ has FIP, then $\widehat{R}(X) = \widehat{\widetilde{R(X)}}$.*

Proof. The map $[R, S] \rightarrow [R(X), S(X)]$ defined by $T \mapsto T(X) = R(X) \otimes_R T$ is bijective [12, Theorem 32], whence $\widehat{\widetilde{R(X)}} = T(X)$ for some $T \in [R, S]$. Moreover, $\widehat{R(X)} \rightarrow \widehat{\widetilde{R(X)}}$ is a flat epimorphism. Since $R \rightarrow R(X)$ is faithfully flat, $\widehat{R} = T$ and the result follows. \square

6. FIBERS OF QUASI-PRÜFER EXTENSIONS

We intend to complete some results of Ayache-Dobbs [5]. We begin by recalling some features about quasi-finite ring morphisms. A ring morphism $R \rightarrow S$ is called quasi-finite by [40] if it is of finite type and $\kappa(P) \rightarrow \kappa(P) \otimes_R S$ is finite (as a $\kappa(P)$ -vector space), for each $P \in \text{Spec}(R)$ [40, Proposition 3, p.40].

Proposition 6.1. *A ring morphism of finite type is incomparable if and only if it is quasi-finite and, if and only if its fibers are finite.*

Proof. Use [41, Corollary 1.8] and the above definition. \square

Theorem 6.2. *An extension $R \subseteq S$ is quasi-Prüfer if and only if $R \subseteq T$ is quasi-finite (respectively, has finite fibers) for each $T \in [R, S]$ such that T is of finite type over R , if and only if $R \subseteq T$ has integral fiber morphisms for each $T \in [R, S]$.*

Proof. It is clear that $R \subseteq S$ is an INC-pair implies the condition because of Proposition 6.1. To prove the converse, let $T \in [R, S]$ and write T as the union of its finite type R -subalgebras T_α . Now let $Q \subseteq Q'$ be prime ideals of T , lying over a prime ideal P of R and set $Q_\alpha := Q \cap T_\alpha$ and $Q'_\alpha := Q' \cap T_\alpha$. If $R \subseteq T_\alpha$ is quasi-finite, then $Q_\alpha = Q'_\alpha$, so that $Q = Q'$ and then $R \subseteq T$ is incomparable. The last statement is Proposition 3.8. \square

Corollary 6.3. *An integrally closed extension is Prüfer if and only if each of its subextensions $R \subseteq T$ of finite type has finite fibers.*

Proof. It is enough to observe that the fibers of a (flat) epimorphism have a cardinal ≤ 1 , because an epimorphism is spectrally injective. \square

A ring extension $R \subseteq S$ is called *strongly affine* if each of its subextensions $R \subseteq T$ is of finite type. The above considerations show that in this case $R \subseteq S$ is quasi-Prüfer if and only if each of its subextensions $R \subseteq T$ has finite fibers. For example, an FCP extension is strongly affine and quasi-Prüfer. We also are interested in extensions $R \subseteq S$ that are not necessarily strongly affine and such that each of its subextensions $R \subseteq T$ have finite fibers.

Next lemma will be useful, its proof is obvious.

Lemma 6.4. *Let $R \subseteq S$ be an extension and $T \in [R, S]$*

- (1) *If $T \subseteq S$ is spectrally injective and $R \subseteq T$ has finite fibers, then $R \subseteq S$ has finite fibers.*
- (2) *If $R \subseteq T$ is spectrally injective, then $T \subseteq S$ has finite fibers if and only if $R \subseteq S$ has finite fibers.*

Remark 6.5. Let $R \subseteq S$ be an almost-Prüfer extension, such that the integral extension $T := \tilde{R} \subseteq S$ has finite fibers and let $P \in \text{Spec}(R)$. The study of the finiteness of $\text{Fib}_{R,S}(P)$ can be reduced as follows. As $\tilde{R} \subseteq S$ is an epimorphism, because it is Prüfer, it is spectrally injective (see Scholium A). The hypotheses of Proposition 4.5 hold. We examine three cases. In case $(R : \tilde{R}) \not\subseteq P$, it is well known that $R_P = (\tilde{R})_P$ so that $|\text{Fib}_{R,S}(P)| = 1$, because $\tilde{R} \rightarrow S$ is spectrally injective. Suppose now that $(R : \tilde{R}) = P$. From $(R : \tilde{R}) = (T : S) \cap R$, we deduce that P is laid over by some $Q \in \text{Spec}(T)$ and then $\text{Fib}_{R,\tilde{R}}(P) \cong \text{Fib}_{T,S}(Q)$. The conclusion follows as above. Thus the remaining case is $(R : \tilde{R}) \subset P$

and we can assume that $PT = T$ for if not $\text{Fib}_{R, \bar{R}}(P) \cong \text{Fib}_{T, S}(Q)$ for some $Q \in \text{Spec}(T)$ by Scholium A (1).

Proposition 6.6. *Let $R \subseteq S$ be an almost-Prüfer extension. If $\tilde{R} \subseteq S$ has finite fiber morphisms and $(\tilde{R}_P : S_P)$ is a maximal ideal of \tilde{R}_P for each $P \in \text{Supp}_R(S/\tilde{R})$, then $R \subseteq \bar{R}$ and $R \subseteq S$ have finite fibers.*

Proof. The Prüfer closure commutes with the localization at prime ideals by Proposition 4.12. We set $T := \tilde{R}$. Let P be a prime ideal of R and $\varphi : R \rightarrow R_P$ the canonical morphism. We clearly have $\text{Fib}_{R, \cdot}(P) = {}^a\varphi(\text{Fib}_{R_P, \cdot}(PR_P))$. Therefore, we can localize the data at P and we can assume that R is local.

In case $(T : S) = T$, we get a factorization $R \rightarrow \bar{R} \rightarrow T$. Since $R \rightarrow T$ is Prüfer so is $R \rightarrow \bar{R}$ and it follows that $R = \bar{R}$ because a Prüfer extension is integrally closed.

From Proposition 1.2 applied to $R \subseteq T$, we get that there is some $\mathfrak{P} \in \text{Spec}(R)$ such that $T = R_{\mathfrak{P}}$, R/\mathfrak{P} is a valuation ring with quotient field T/\mathfrak{P} and $\mathfrak{P} = \mathfrak{P}T$. It follows that $(T : S) = \mathfrak{P}T = \mathfrak{P} \subseteq R$, and hence $(T : S) = (T : S) \cap R = (R : \bar{R})$. We have therefore a pushout diagram by Theorem 4.6:

$$\begin{array}{ccc} R' := R/\mathfrak{P} & \longrightarrow & \bar{R}'/\mathfrak{P} := \bar{R}' \\ \downarrow & & \downarrow \\ T' := T/\mathfrak{P} & \longrightarrow & S/\mathfrak{P} := S' \end{array}$$

where R/\mathfrak{P} is a valuation domain, T/\mathfrak{P} is its quotient field and $\bar{R}'/\mathfrak{P} \rightarrow S/\mathfrak{P}$ is Prüfer by [26, Proposition 5.8, p. 52].

Because $\bar{R}' \rightarrow S'$ is injective and a flat epimorphism, there is a bijective map $\text{Min}(S') \rightarrow \text{Min}(\bar{R}')$. But $T' \rightarrow S'$ is the fiber at \mathfrak{P} of $T \rightarrow S$ and is therefore finite. Therefore, $\text{Min}(S')$ is a finite set $\{N_1, \dots, N_n\}$ of maximal ideals lying over the minimal prime ideals $\{M_1, \dots, M_n\}$ of \bar{R}' lying over 0 in R' . We infer from Lemma 3.7 that $\bar{R}'/M_i \rightarrow S'/N_i$ is Prüfer, whence integrally closed. Therefore, \bar{R}'/M_i is an integral domain and the integral closure of R' in S'/N_i . Any maximal ideal M of \bar{R}' contains some M_i . To conclude it is enough to use a result of Gilmer [19, Corollary 20.3] because the number of maximal ideals in \bar{R}'/M_i is less than the separable degree of the extension of fields $T' \subseteq S'/N_i$. \square

Remark 6.7. (1) Suppose that $(\tilde{R} : S)$ is a maximal ideal of \tilde{R} . We clearly have $(\tilde{R} : S)_P \subseteq (\tilde{R}_P : S_P)$ and the hypotheses on $(\tilde{R} : S)$ of the above proposition hold.

(2) In case $\tilde{R} \subseteq S$ is a tower of finitely many integral minimal extensions $R_{i-1} \subseteq R_i$ with $M_i = (R_{i-1} : R_i)$, then $\text{Supp}_{\tilde{R}}(S/\tilde{R}) = \{N_1, \dots, N_n\} \subseteq \text{Max}(\tilde{R})$ where $N_i = M_i \cap R$. If the ideals N_i are different, each localization at N_i of $\tilde{R} \subseteq S$ is integral minimal and the above result may apply. This generalizes the Ayache-Dobbs result [5, Lemma 3.6], where $\tilde{R} \subseteq S$ is supposed to be integral minimal.

Proposition 6.8. *Let $R \subseteq S$ be a quasi-Prüfer ring extension.*

- (1) *$R \subseteq S$ has finite fibers if and only if $R \subseteq \bar{R}$ has finite fibers.*
- (2) *$R \subseteq \bar{R}$ has finite fibers if and only if each extension $R \subseteq T$, where $T \in [R, S]$ has finite fibers.*

Proof. (1) Let $P \in \text{Spec}(R)$ and the morphisms $\kappa(P) \rightarrow \kappa(P) \otimes_R \bar{R} \rightarrow \kappa(P) \otimes_R S$. The first (second) morphism is integral (a flat epimorphism) because deduced by base change from the integral morphism $R \rightarrow \bar{R}$ (the flat epimorphism $\bar{R} \rightarrow S$). Therefore, the ring $\kappa(P) \otimes_R \bar{R}$ is zero dimensional, so that the second morphism is surjective by Scholium A (2). Set $A := \kappa(P) \otimes_R \bar{R}$ and $B := \kappa(P) \otimes_R S$, we thus have a module finite flat ring morphism $A \rightarrow B$. Hence, $A_Q \rightarrow B_Q$ is free for each $Q \in \text{Spec}(A)$ [16, Proposition 9] and $B_Q \neq 0$ because it contains $\kappa(P) \neq 0$. Therefore, $A_Q \rightarrow B_Q$ is injective and it follows that $A \cong B$.

(2) Suppose that $R \subseteq \bar{R}$ has finite fibers and let $T \in [R, S]$, then $\bar{R} \subseteq \bar{R}T$ is a flat epimorphism by Proposition 4.5(1) and so is $\kappa(P) \otimes_R \bar{R} \rightarrow \kappa(P) \otimes_R \bar{R}T$. Since $\text{Spec}(\kappa(P) \otimes_R \bar{R}T) \rightarrow \text{Spec}(\kappa(P) \otimes_R \bar{R})$ is injective, $R \subseteq \bar{R}T$ has finite fibers. Now $R \subseteq T$ has finite fibers because $T \subseteq \bar{R}T$ is integral and is therefore spectrally surjective. \square

Remark 6.9. Actually, the statement (1) is valid if we only suppose that $\bar{R} \subseteq S$ is a flat epimorphism.

Next result contains [5, Lemma 3.6], gotten after a long proof.

Corollary 6.10. *Let $R \subseteq S$ be an almost-Prüfer extension. Then $R \subseteq S$ has finite fibers if and only if $R \subseteq \bar{R}$ has finite fibers, and if and only if $\tilde{R} \subseteq S$ has finite fibers.*

Proof. By Proposition 6.8(1) the first equivalence is clear. The second is a consequence of Lemma 6.4(2). \square

The following result is then clear.

Theorem 6.11. *Let $R \subseteq S$ be a quasi-Prüfer extension with finite fibers, then $R \subseteq T$ has finite fibers for each $T \in [R, S]$.*

Corollary 6.12. *If $R \subseteq S$ is quasi-finite and quasi-Prüfer, then $R \subseteq T$ has finite fibers for each $T \in [R, S]$ and $\tilde{R} \subseteq S$ is module finite.*

Proof. By the Zariski Main Theorem, there is a factorization $R \subseteq F \subseteq S$ where $R \subseteq F$ is module finite and $F \subseteq S$ is a flat epimorphism [40, Corollaire 2, p.42]. To conclude, we use Scholium A in the rest of the proof. The map $\tilde{R} \otimes_R F \rightarrow S$ is injective because $F \rightarrow \tilde{R} \otimes_R F$ is a flat epimorphism and is surjective, since it is integral and a flat epimorphism because $\tilde{R} \otimes_R F \rightarrow S$ is a flat epimorphism. \square

Corollary 6.13. *An FMC extension $R \subseteq S$ is such that $R \subseteq T$ has finite fibers for each $T \in [R, S]$.*

Proof. Such an extension is quasi-finite and quasi-Prüfer. Then use Corollary 6.12. \square

[5, Example 4.7] exhibits some FMC extension $R \subseteq S$, such that $R \subseteq \bar{R}$ has not FCP. Actually, $[R, \bar{R}]$ is an infinite (maximal) chain.

Proposition 6.14. *Let $R \subseteq S$ be a quasi-Prüfer extension such that $R \subseteq \bar{R}$ has finite fibers and R is semi-local. Then T is semi-local for each $T \in [R, S]$.*

Proof. Obviously \bar{R} is semi-local. From the tower $\bar{R} \subseteq T\bar{R} \subseteq S$ we deduce that $\bar{R} \subseteq T\bar{R}$ is Prüfer. It follows that $T\bar{R}$ is semi-local [5, Lemma 2.5 (f)]. As $T \subseteq T\bar{R}$ is integral, we get that T is semi-local. \square

The following proposition gives a kind of converse.

Proposition 6.15. *Let $R \subseteq S$ be an extension with \bar{R} semi-local. Then $R \subseteq S$ is quasi-Prüfer if and only if T is semi-local for each $T \in [R, S]$.*

Proof. If $R \subseteq S$ is quasi-Prüfer, $\bar{R} \subseteq S$ is Prüfer. Let $T \in [R, S]$ and set $T' := T\bar{R}$, so that $T \subseteq T'$ is integral, and $\bar{R} \subseteq T'$ is Prüfer (and then a normal pair). It follows from [5, Lemma 2.5 (f)] that T' is semi-local, and so is T .

If T is semi-local for each $T \in [R, S]$, so is any $T \in [\bar{R}, S]$. Then, (\bar{R}, S) is a residually algebraic pair [6, Theorem 3.10] (generalized to arbitrary extensions) and so is $\bar{R}_M \subseteq S_M$ for each $M \in \text{Max}(\bar{R})$, whence is Prüfer [6, Theorem 2.5] (same remark) and Proposition 1.2. Then, $\bar{R} \subseteq S$ is Prüfer by Proposition 1.1 and $R \subseteq S$ is quasi-Prüfer. \square

7. NUMERICAL PROPERTIES OF FCP EXTENSIONS

Lemma 7.1. *Let $R \subset S$ be an FCP extension. The map $\varphi : [R, S] \rightarrow \{(T', T'') \in [R, \bar{R}] \times [\bar{R}, S] \mid \text{Supp}_{T'}(\bar{R}/T') \cap \text{Supp}_{T''}(T''/\bar{R}) = \emptyset\}$, defined by $\varphi(T) := (T \cap \bar{R}, \bar{R}T)$ for each $T \in [R, S]$, is bijective. In particular, if $R \subset S$ has FIP, then $|[R, S]| \leq |[R, \bar{R}]||[\bar{R}, S]|$.*

Proof. Let $(T', T'') \in [R, \overline{R}] \times [\overline{R}, S]$. Then, \overline{R} is also the integral closure of T' in T'' (and in S).

Let $T \in [R, S]$. Set $T' := T \cap \overline{R}$ and $T'' := \overline{R}T$. Then $(T', T'') \in [R, \overline{R}] \times [\overline{R}, S]$. Assume that $T' = T''$, so that $T' = T'' = \overline{R}$, giving $T = \overline{R}$ and $\text{Supp}_{T'}(\overline{R}/T') = \text{Supp}_{T''}(T''/\overline{R}) = \emptyset$. Assume that $T' \neq T''$. In view of [39, Proposition 3.6], we get $\text{Supp}_{T'}(\overline{R}/T') \cap \text{Supp}_{T''}(T''/\overline{R}) = \emptyset$. Hence φ is well defined.

Now, let $T_1, T_2 \in [R, S]$ be such that $\varphi(T_1) = \varphi(T_2) = (T', T'')$. Assume $T' \neq T''$. Another use of [39, Proposition 3.6] gives that $T_1 = T_2$. If $T' = T''$, then, $T' = T'' = \overline{R}$, so that $T_1 = T_2 = \overline{R}$. It follows that φ is injective. The same reference gives that φ is bijective. \square

Proposition 7.2. *Let $R \subset S$ be a FCP extension. We define two order-isomorphisms φ' and ψ as follows:*

$$\begin{aligned} \varphi' : [R, \tilde{R}] &\rightarrow [R, \overline{R}] \times [\overline{R}, \tilde{R}] \text{ defined by } \varphi'(T) := (T \cap \overline{R}, T\overline{R}) \\ \psi : [R, \tilde{R}] &\rightarrow [R, \tilde{R}] \times [\tilde{R}, \tilde{R}] \text{ defined by } \psi(T) := (T \cap \tilde{R}, T\tilde{R}). \end{aligned}$$

Proof. This follows from [39, Lemma 3.7] and Proposition 4.19. (We recall that $\tilde{R} = \overline{\tilde{R}}$.) \square

Corollary 7.3. *If $R \subseteq S$ has FCP, then $\text{Supp}(\tilde{R}/\tilde{R}) = \text{Supp}(\overline{R}/R)$, $\text{Supp}(\tilde{R}/\overline{R}) = \text{Supp}(\tilde{R}/R)$ and $\text{Supp}(\tilde{R}/R) = \text{Supp}(\tilde{R}/R) \cup \text{Supp}(\overline{R}/R)$.*

Proof. Set $A := \text{Supp}(\overline{R}/\tilde{R})$, $B := \text{Supp}(\tilde{R}/R)$, $C := \text{Supp}(\overline{R}/\overline{R})$ and $D := \text{Supp}(\overline{R}/R)$. Then, $A \cup B = C \cup D = \text{Supp}(\overline{R}/R)$, with $A \cap B = C \cap D = B \cap D = \emptyset$ by Proposition 4.19.

Assume that $A \cup B \neq B \cup D$ and let $P \in (A \cup B) \setminus (B \cup D)$. Then, $R_P \neq (\tilde{R})_P = (\overline{R})_P(\tilde{R})_P = R_P$, a contradiction. It follows that $A \cup B = B \cup D$. Intersecting the two members of this equality with A and D , we get $A = A \cap D = D$. In the same way, intersecting the equality $A \cup B = C \cup D = C \cup A$ by B and C , we get $B = C$. \square

Corollary 7.4. *Let $R \subset S$ be an FCP extension. We define two order-isomorphisms*

$$\begin{aligned} \varphi_1 : [R, \tilde{R}] &\rightarrow [\overline{R}, \tilde{R}] \text{ by } \varphi_1(T) := T\overline{R} \\ \psi_1 : [R, \overline{R}] &\rightarrow [\tilde{R}, \tilde{R}] \text{ by } \psi_1(T) := T\tilde{R}. \end{aligned}$$

Proof. We use notation of Proposition 4.19. We begin to remark that \overline{R} and \tilde{R} play symmetric roles.

Let $T, T' \in [R, \tilde{R}]$ be such that $\varphi_1(T) = \varphi_1(T')$. Since $T \cap \overline{R} = T' \cap \overline{R} = R$ by Proposition 4.19, we get $\varphi(T) = \varphi(T')$, so that $T = T'$ and φ_1 is injective. A similar argument shows that ψ_1 is injective.

Let $U \in [\overline{R}, \widetilde{R}]$. There exists $T \in [R, \widetilde{R}]$ such that $\varphi(T) = (R, U)$, so that $R = T \cap \overline{R}$ and $U = T\overline{R}$. Let $M \in \text{Supp}(\widetilde{R}/R) = \text{Supp}(\widetilde{R}/R) \cup \text{Supp}(\overline{R}/R)$ by Corollary 7.3. If $M \in \text{Supp}(\widetilde{R}/R)$, then $M \notin \text{Supp}(\overline{R}/R)$ by Proposition 4.19, giving $T_M \subseteq \widetilde{R}_M = \overline{R}_M \widetilde{R}_M = \widetilde{R}_M$. If $M \in \text{Supp}(\overline{R}/R)$, the same reasoning gives $T_M \subseteq \overline{R}_M$, so that $R_M = T_M \cap \overline{R}_M = T_M$, but $R_M = \widetilde{R}_M$. Then, $T_M = \widetilde{R}_M$. It follows that $T \subseteq \widetilde{R}$, giving $T \in [R, \widetilde{R}]$ and φ_1 is surjective, hence bijective. A similar argument shows that ψ_1 is surjective, hence bijective. \square

Corollary 7.5. *If $R \subset S$ has FCP, then $\theta : [R, \widetilde{R}] \times [R, \overline{R}] \rightarrow [R, \widetilde{R}]$ defined by $\theta(T, T') := TT'$, is an order-isomorphism. In particular, if $R \subset S$ has FIP, then $|[R, \widetilde{R}]||[R, \overline{R}]| = |[R, \widetilde{R}]| \leq |[R, S]|$.*

Proof. Using notation of Proposition 7.2 and Corollary 7.4, we may remark that $\psi \circ \theta = \text{Id} \times \psi_1$. Since ψ and $\text{Id} \times \psi_1$ are order-isomorphisms, so is θ . The FIP case is obvious. \square

Gathering the previous results, we get the following theorem.

Theorem 7.6. *If $R \subset S$ has FCP, the next statements are equivalent:*

- (1) $\text{Supp}(\overline{R}/R) \cap \text{Supp}(S/\overline{R}) = \emptyset$.
- (2) *The map $\varphi : [R, S] \rightarrow [R, \overline{R}] \times [\overline{R}, S]$ defined by $\varphi(T) := (T \cap \overline{R}, T\overline{R})$ is an order-isomorphism.*
- (3) $R \subseteq S$ is almost-Prüfer.
- (4) $\text{Supp}(S/\overline{R}) = \text{Supp}(\widetilde{R}/R)$.
- (5) *The map $\varphi_1 : [R, \widetilde{R}] \rightarrow [\overline{R}, S]$ defined by $\varphi_1(T) := T\overline{R}$ is an order-isomorphism.*
- (6) *The map $\psi_1 : [R, \overline{R}] \rightarrow [\widetilde{R}, S]$ defined by $\psi_1(T) := T\widetilde{R}$ is an order-isomorphism.*
- (7) *The map $\theta : [R, \widetilde{R}] \times [R, \overline{R}] \rightarrow [R, S]$ defined by $\theta(T, T') := TT'$ is an order-isomorphism.*

If one of these conditions holds, then $\text{Supp}(S/\widetilde{R}) = \text{Supp}(\overline{R}/R)$.

If $R \subset S$ has FIP, the former conditions are equivalent to each of the following conditions:

- (8) $|[R, S]| = |[R, \widetilde{R}]||[R, \overline{R}]|$.
- (9) $|[R, S]| = |[R, \overline{R}]||[\overline{R}, S]|$.
- (10) $|[R, \widetilde{R}]| = |[\overline{R}, S]|$.
- (11) $|[R, \overline{R}]| = |[\widetilde{R}, S]|$.

Proof. (1) \Rightarrow (2) by [39, Lemma 3.7].

(2) \Rightarrow (1). If the statement (2) holds, there exists $T \in [R, S]$ such that $T \cap \overline{R} = R$ and $T\overline{R} = S$. Then, [39, Proposition 3.6] gives that $\text{Supp}(\overline{R}/R) \cap \text{Supp}(S/\overline{R}) = \emptyset$.

(1) \Rightarrow (3) by [39, Proposition 3.6].

(3) \Rightarrow (4), (5), (6) and (7): Use Corollary 7.3 to get (4), Corollary 7.4 to get (5) and (6), and Corollary 7.5 to get (7). Moreover, (3) and Corollary 7.3 give $\text{Supp}(S/\tilde{R}) = \text{Supp}(\overline{R}/R)$.

(4) \Rightarrow (1) by Proposition 4.19(2).

(5), (6) or (7) \Rightarrow (3) because, in each case, we have $S = \overline{R}\tilde{R}$.

Assume now that $R \subset S$ has FIP.

Then, obviously, (7) \Rightarrow (8), (2) \Rightarrow (9), (5) \Rightarrow (10) and (6) \Rightarrow (11).

(9) \Rightarrow (3) by Corollary 7.5, which gives $|[R, \tilde{R}]||[R, \overline{R}]| = |[R, \tilde{R}]|$, so that $|[R, S]| = |[R, \tilde{R}]|$, and then $S = \tilde{R}$.

(8) \Rightarrow (1): Using the map φ of Lemma 7.1, we get that $\{(T', T'') \in [R, \overline{R}] \times [\tilde{R}, S] \mid \text{Supp}_{T'}(\overline{R}/T') \cap \text{Supp}_{T''}(T''/\overline{R}) = \emptyset\} = [R, \overline{R}] \times [\tilde{R}, S]$, so that $\text{Supp}_R(\overline{R}/R) \cap \text{Supp}_R(S/\overline{R}) = \emptyset$.

(10) \Rightarrow (3) and (11) \Rightarrow (3) by Corollary 7.4. \square

Example 7.7. We give an example where the results of Theorem 7.6 do not hold if $R \subseteq S$ has not FCP. Set $R := \mathbb{Z}_P$ and $S := \mathbb{Q}[X]/(X^2)$, where $P \in \text{Max}(\mathbb{Z})$. Then, $\tilde{R} = \mathbb{Q}$ because $R \subset \tilde{R}$ is Prüfer (minimal) and $\tilde{R} \subset S$ is integral minimal. Set $M := PR_P \in \text{Max}(R)$ with (R, M) a local ring. It follows that $M \in \text{Supp}(\overline{R}/R) \cap \text{Supp}(S/\overline{R})$ because $R \subset S$ is neither integral, nor Prüfer. Similarly, $M \in \text{Supp}(\overline{R}/R) \cap \text{Supp}(\tilde{R}/R)$. Indeed, $R \subset \overline{R}$ has not FCP.

We end the paper by some length computations in the FCP case.

Proposition 7.8. *Let $R \subseteq S$ be an FCP extension. The following statements hold:*

- (1) $\ell[R, \tilde{R}] = \ell[\overline{R}, \tilde{R}]$ and $\ell[R, \overline{R}] = \ell[\tilde{R}, \tilde{R}]$
- (2) $\ell[R, \tilde{R}] = \ell[R, \tilde{R}] + \ell[\tilde{R}, \tilde{R}] = \ell[R, \overline{R}] + \ell[\overline{R}, \tilde{R}]$
- (3) $\ell[\overline{R}, \tilde{R}] = |\text{Supp}_{\overline{R}}(\tilde{R}/\overline{R})| = \ell[R, \tilde{R}] = |\text{Supp}_R(\tilde{R}/R)|$.

Proof. To prove (1), use the maps φ_1 and ψ_1 of Corollary 7.4. Then (2) follows from [11, Theorem 4.11] and (3) from [10, Proposition 6.12]. \square

REFERENCES

- [1] D.F. Anderson and D.E. Dobbs, Pairs of rings with the same prime ideals, *Can. J. Math.* **XXXII**, (1980), 362–384.
- [2] D. F. Anderson, D. E. Dobbs and M. Fontana, On treed Nagata rings, *J. Pure Appl. Algebra*, **61**, (1989), 107–122.

- [3] A. Ayache, M. Ben Nasr, O. Echi and N. Jarboui, Universally catenarian and going-down pairs of rings, *Math. Z.*, **238**, (2001), 695–731.
- [4] A. Ayache, A constructive study about the set of intermediate rings, *Comm. Algebra*, **41** (2013), 4637–4661.
- [5] A. Ayache and D. E. Dobbs, Finite maximal chains of commutative rings, *JAA*, **14**, (2015), 14500751–1450075-27.
- [6] A. Ayache and A. Jaballah, Residually algebraic pairs of rings, *Math. Z.*, **225**, (1997), 49–65.
- [7] M. Ben Nasr and N. Jarboui, New results about normal pairs of rings with zero-divisors, *Ricerche mat.* **63** (2014), 149–155.
- [8] R.D. Chatham, Going-down pairs of commutative rings, *Rendiconti del Circolo Matematico di Palermo, Serie II, Tomo L*, (2001), 509–542.
- [9] G. W. Chang and M. Fontana, Uppers to 0 in polynomial rings and Prüfer-like domains, *Comm. Algebra*, **37** (2009), 164–192.
- [10] D. E. Dobbs, G. Picavet and M. Picavet-L’Hermitte, Characterizing the ring extensions that satisfy FIP or FCP, *J. Algebra*, **371** (2012), 391–429.
- [11] D. E. Dobbs, G. Picavet and M. Picavet-L’Hermitte, Transfer results for the FIP and FCP properties of ring extensions, *Comm. Algebra*, **43** (2015), 1279–1316.
- [12] D. E. Dobbs, G. Picavet and M. Picavet-L’Hermitte, *When an extension of Nagata rings has only finitely many intermediate rings, each of those is a Nagata ring?*, *Int. J. Math. Math. Sci.*, **2014** (2014), Article ID315919, 13 pp.
- [13] D. E. Dobbs, G. Picavet, M. Picavet-L’Hermitte and J. Shapiro, On intersections and composites of minimal ring extensions, *J P J. Algebra, Number Theory Appl.*, **26** (2012), 103–158.
- [14] D. E. Dobbs, On characterizations of integrality involving the lying-over and incomparability properties, *J. Comm. Algebra*, **1** (2009), 227–235.
- [15] D.E. Dobbs and J. Shapiro, Pseudo-normal pairs of integral domains, *Houston J. Math.*, **40** (2014), 1–9.
- [16] S. Endo, On semi-hereditary rings, *J. Math. Soc. Japan*, **13** (1961), 109–119.
- [17] D. Ferrand and J.-P. Olivier, Homomorphismes minimaux d’anneaux, *J. Algebra*, **16** (1970), 461–471.
- [18] M. Fontana, J. A. Huckaba and I. J. Papick, Prüfer domains, Dekker, New York, 1997.
- [19] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.
- [20] M. Griffin, Prüfer rings with zero divisors, *Journal für die reine und angewandte Mathematik*, **239**, (1969), 55–67.
- [21] M. Grandet, Une caractérisation des morphismes minimaux non entiers, *C.R. Acad. Sc. Paris*, **271**, (1970), Série A 581–583.
- [22] A. Grothendieck and J. Dieudonné, *Eléments de Géométrie Algébrique*, Springer Verlag, Berlin, (1971).
- [23] E. Houston, Uppers to zero in polynomial rings, pp. 243–261, in: *Multiplicative Ideal Theory in Commutative Algebra*, Springer-Verlag, New York, 2006.
- [24] A. Jaballah, Finiteness of the set of intermediary rings in normal pairs, *Saitama Math. J.*, **17** (1999), 59–61.

- [25] N. Jarboui and E. Massoud, On finite saturated chains of overrings, *Comm. Algebra*, **40**, (2012), 1563–1569.
- [26] M. Knebusch and D. Zhang, Manis Valuations and Prüfer Extensions I, Springer, Berlin, 2002.
- [27] D. Lazard, Autour de la platitude, *Bull. Soc. Math. France*, **97**, (1969), 81–128.
- [28] M. Lazarus, Fermeture intégrale et changement de base, *Ann. Fac. Sci. Toulouse*, **6**, (1984), 103–120.
- [29] T. G. Lucas, Some results on Prüfer rings, *Pacific J. Math.*, **124**, (1986), 333–343.
- [30] K. Morita, Flat modules, Injective modules and quotient rings, *Math. Z.*, **120** (1971), 25–40.
- [31] J.P. Olivier, Anneaux absolument plats universels et épimorphismes à buts réduits, *Séminaire Samuel. Algèbre Commutative*, Tome 2 (1967-1968), exp. n° 6, p. 1–12.
- [32] J. P. Olivier, Montée des propriétés par morphismes absolument plats, *J. Alg. Pure Appl.*, Université des Sciences et Technique du Languedoc, Montpellier, France (1971).
- [33] J.P. Olivier, Going up along absolutely flat morphisms, *J. Pure Appl. Algebra*, **30** (1983), 47–59.
- [34] G. Picavet, Propriétés et applications de la notion de contenu, *Comm. Algebra*, **13**, (1985), 2231–2265.
- [35] G. Picavet, Universally going-down rings, 1-split rings, and absolute integral closure, *Comm. Algebra*, **31**, (2003), 4655–4685.
- [36] G. Picavet and M. Picavet-L’Hermitte, About minimal morphisms, pp. 369–386, in: *Multiplicative Ideal Theory in Commutative Algebra*, Springer, New York, 2006.
- [37] G. Picavet, Seminormal or t-closed schemes and Rees rings, *Algebra Repr. Theory*, **1**, (1998), 255–309.
- [38] G. Picavet and M. Picavet-L’Hermitte, Some more combinatorics results on Nagata extensions, *Palestine J. Math.*, **1**, (**Spec.1**), (2016), 49–62.
- [39] G. Picavet and M. Picavet-L’Hermitte, Prüfer and Morita hulls of FCP extensions, *Comm. Algebra*, **43**, (2015), 102–119.
- [40] M. Raynaud, Anneaux locaux Henséliens, *Lect. Notes in Math.*, Springer, Vol. 169, (1970).
- [41] H. Uda, Incomparability in ring extensions, *Hiroshima Math. J.*, **9**, (1979), 451–463.
- [42] S. Visweswaran, Laskerian pairs, *J. Pure Appl. Algebra*, **59**, (1989), 87–110.

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